Chapter 11: Voting and Apportionment

Figure 1: Voters cast their ballots in one of the world's many democracies.

Suppose a friend asked you, "When did you last vote?" What would your answer be? Maybe you would tell your friend that the last time you voted was during the last presidential election, or perhaps you would tell your friend that you prefer not to vote. When thinking about voting, presidential campaigns or advertisements for reelections may come to mind, but you can cast your vote in many ways. Have you liked a post, followed a creator, friended a stranger, or clicked a heart online today? In the digital age, it's possible to vote several times a day. Voting systems are not only the machines that drive every democracy on Earth, but they are also the engines driving social media and many other aspects of life. A deeper understanding of these voting systems will enhance your ability to successfully engage with the world in which we live.

In this chapter, you will become one of the founders of the new democratic country of Imaginaria. You have a great responsibility to the people of this fledgling democracy. You have been tasked with writing the portion of the constitution that lays out voting procedures. In preparation for this important task, you will explore the various types of voting systems, from school board elections to Twitter wars. You will see how these types are alike, how they differ, and how they might be applied in Imaginaria. Most importantly, you will learn about the mathematically inherent advantages and disadvantages of various voting systems so that you can make informed choices to better the lives of the Imaginarians.

11.1 Voting Methods

Figure 2: President Barack Obama votes in the 2012 election.

Learning Objectives

After completing this section, you should be able to:

1. Apply plurality voting to determine a winner.
2. Apply runoff voting to determine a winner.
3. Apply ranked-choice voting to determine a winner.
4. Apply Borda count voting to determine a winner.
5. Apply pairwise comparison and Condorcet voting to determine a winner.
6. Apply approval voting to determine a winner.
7. Compare and contrast voting methods to identify flaws.

Today is the day that you begin your quest to collaborate on the constitution of Imaginaria! Let's begin by thinking about the selection of a leader who can serve as president. It seems straightforward; if the majority of citizens prefer a particular candidate, that candidate should win. But not all votes are decided by a simple majority. Why not? What are the options?

Majority versus Plurality Voting

When an election involves only two options, a simple **majority** is a reasonable way to determine a winner. A majority is a number equaling more than half, or greater than 50 percent of the total.

Let's take a look at the outcomes of U.S. presidential elections to understand more. displays the results of the 2000 U.S. presidential election. Like most presidential elections, this election involved more than two options. If that is the case, is it possible that no single candidate will receive more than half of the votes cast?

Candidate (Party Label)	Popular Vote Total
Al Gore (Democrat)	50,999,897
George W. Bush (Republican)	50,456,002
Ralph Nader (Green)	2,882,955
Patrick J. Buchanan (Reform/Independent)	448,895
Harry Browne (Libertarian)	384,431
Howard Phillips (Constitution)	98,020
Other	134,900
Total:	**105,405,100**

Example 1 **Majority of Popular Vote in the 2000 U.S. Presidential Election**

Refer back to . Did any single candidate secure the majority of popular votes?

Solution

Step 1: Calculate 50 percent of 105,405,100 by multiplying the decimal form of 50 percent, which is 0.50, by 158,394,605: $0.50(105,405,100) = 52,702,550$

Step 2: Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 52,702,550; so, a majority candidate must have 52,702,551 votes or more.

Step 3: Compare the number of votes each candidate received to 52,702,551. According to the data in , none of the candidates secured a majority.

Unlike in the 2000 U.S. presidential election, a candidate won the majority of votes in the 2020 election. It is a common occurrence for no single candidate to receive a majority of the votes in an election with more than two candidates. When this occurs, the candidate with the largest portion of the votes is said to have a **plurality**.

Example 2 **Plurality of Popular Vote in the 2000 U.S. Presidential Election**

Refer again to . In the 2000 U.S. presidential election, which candidate had a plurality of popular votes?

Solution

Al Gore secured 50,999,897 votes which was more than any other single candidate. Therefore, he had a plurality of the popular votes.

Your plans for Imaginarian elections will likely include primary elections, or preliminary elections to select candidates for a principal or general election. displays the results of the 2018 U.S. Senate Republican primary for Maryland.

Top Four Republican Candidates	Votes	Percentage of Party Votes
Cambell, Tony	51,426	29.22%
Chaffee, Chris	42,328	24.05%
Grigorian, Christina J.	30,756	17.48%
Graziani, John R.	15,435	8.77%
Total Votes	175,981	100%

Consider how election by plurality, not majority, is the most common method of selecting candidates for public office.

WHO KNEW?

The U.S. Electorial College: Winning the Presidency without a Plurality

In the 2000 U.S. presidential election, Al Gore had a plurality of the popular votes, but he did not win the election. Why? This occurred because the U.S. president and vice president are elected by electors rather than a direct vote by the citizens. The electors are part of the Electoral College, a body of people representing the states. Why was the Electoral College created? The Electoral College was created as a compromise between those authors of the U.S. Constitution who believed Congress should elect the president, and those who believed the citizens should vote directly. The popular vote was not recorded until the presidential election of 1824. Since then, only five presidents have been elected without winning a plurality of the popular vote: John Quincy Adams in 1824, Rutherford B. Hayes in 1876, Benjamin Harrison in 1888, George Bush in 2000, and Donald Trump in 2016.

Runoff Voting

Has your family ever debated what to have for dinner? Suppose your family is deciding on a restaurant and exactly half of you want to have pizza but the other half want hamburgers. How do you decide when the result is a tie? You need a tiebreaker!

Will the new democracy of Imaginaria need tiebreakers? When no candidate satisfies the requirements to win the election, a **runoff election**, or second election, is held to determine a winner.

How would runoff voting work in Imaginaria? There are many types of **runoff voting systems**, which are voting systems that utilize a runoff election when the first round does not result in a winner. The method for implementing a runoff election can vary widely, particularly in the criteria used to determine whether a candidate will be on the ballot in the second election. For example, a **two-round system** is a runoff voting system in which only the top candidates advance to the runoff election. In some two-round systems, only the top two candidates are on the second ballot, or it may be any candidate who secures a certain percentage of the vote will advance. The **Hare Method** is another runoff voting system in which only the candidate(s) with the very least votes are eliminated. This can potentially result in several rounds of runoff elections.

Example 3 **Runoff Election for Condominium Association President**

A condominium association elects a new president every two years by a two-round system of voting. If none of the candidates receive a majority, the association charter states that the top two candidates will be eligible to participate in a runoff election. In a particular year, five residents were nominated. The results of the first round are given in the table below.

Candidate	Votes in First Round
Abou	18
Baiocchi	10
Campana	5
Dali	11
Eugene	4

1. Is there a winner based on the first round? Why or why not?
2. If there is a winner, who won? If there should be a runoff, who will advance to the second round?

Solution

1. A majority of 48 total votes is required to win. Begin by finding 50 percent of 48, which is calculated as follows: $0.50(48) = 24$. A majority is 25 or more. No candidate has a majority, so there is no winner based on the first round.
2. Abou and Dali advance to the second round.

Steps to Determine Winner by Plurality or Majority Election with Runoff

To determine the winner by plurality or when a majority election with runoff occurs, we take these three steps:

Step 1: If a majority is required to win the election, determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes. If a majority is not required, move to Step 2.

Step 2: Count the number of votes for each candidate in the current round of voting. If a single candidate has enough votes to win a plurality, or a majority as appropriate, then you are done! Otherwise, eliminate a predetermined number of candidates based on the rules of the election. Elimination conditions may vary. For example, the rules may state that the candidate(s) with the fewest votes will be eliminated (as in the Hare method), or that only the candidates meeting a certain threshold will move on (as in a two-round system). Once the appropriate candidates are eliminated, move on to Step 3.

Step 3: Hold a runoff election. If the runoff is simulated using a list of voter's preferences, renumber the preferences to reflect the remaining number of options in such a way that the original order of preference is retained. Then repeat Step 2.

Note: The second and third steps may be repeated as many times as necessary for voting procedures that allow multiple runoffs.

Example 4 **Family Dinner Night**

The five members of the Chionilis family—Annette, Rene, Seema, Titus, and Galen—have decided to get takeout for dinner. They are trying to decide on a restaurant. The options are Rainbow China, Dough Boys Pizza, Taco City, or Caribbean Flavor. They will use majority election with runoffs where the restaurant with the fewest votes is eliminated in each round. The preferences of each family member are listed by first initial in the table below. An entry of 1 represents the person's first choice; 2, their second; and so on. For example, Annette's second choice is Dough Boys Pizza.

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Taco City	3	4	2	4	2
Caribbean Flavor	4	1	4	3	4

Use the information in the table to answer the following questions.

1. Which common type of runoff voting method is this?
2. List the results of each round of voting based on this information and determine which restaurant was ultimately chosen.

Solution

1. the Hare Method
2. **Step 1:** Determine the number of votes necessary to have a majority. There are five family members, so 50 percent of 5 is $0.50(5) = 2.5$. A majority is three or more votes.
   Step 2: Count the number of votes for each restaurant in the first election. In a list of voter preferences, the 1s represent the top choice of each voter, which corresponds to their vote in the first round.
   Results of Round 1:
   - Rainbow China — 2 votes

- Dough Boys — 2 votes
- Taco City — 0 votes
- Caribbean Flavor — 1 vote

No restaurant received a majority. Eliminate Taco City, which has the fewest first place votes:

Options	A	R	S	T	G
Rainbow China	1	3	3	1	3
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	4	1	4	3	4

Step 3: Hold a runoff election. In other words, hold a second round. Since we have a list of the voters' preferences with the eliminated option removed, we will renumber the preferences as first, second, and third so that we keep the original order of preference. The result is that we will count the second-place vote of any voter whose first choice was eliminated.

Options	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1
Caribbean Flavor	3	1	3	3	3

Step 4: Repeat the process from Step 2. Count the number votes for each restaurant in the first-round election. Since we are using a list of preferences, we need to count the number of 1s received by each restaurant.
Results of Round 2:

- Rainbow China — 2 votes
- Dough Boys — 2 votes
- Caribbean Flavor — 1 vote

No single restaurant has three votes. Eliminate Caribbean Flavor, which has the fewest first place votes.

OPTIONS	A	R	S	T	G
Rainbow China	1	3	2	1	2
Dough Boys Pizza	2	2	1	2	1

Step 5: Repeat the process from Step 3. Hold another runoff election. This will be Round 3. Renumber the voters' preferences as first and second this time.

Options	A	R	S	T	G
Rainbow China	1	2	2	1	2
Dough Boys Pizza	2	1	1	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each remaining restaurant. Results of Round 3:

- Rainbow China — 2 votes
- Dough Boys — 3 votes

Determine whether any one choice has a majority. Yes! Dough Boys has three votes, so it is the winner!

Ranked-Choice Voting

In Example 4 and Your Turn 11.4, you were given a list that ordered each voter's preferences. This ordering is called a **preference ranking**. A ballot in which a voter is required to give an ordering of their preferences is a **ranked ballot**, and any voting system in which a voter uses a ranked ballot is referred to as **ranked voting**.

The vote for the Academy Awards uses a ranked ballot. The table below provides an example of a ranked ballot for the 2020 Academy Award nominees for Best Director.

Candidate for Best Director	Rank top choice as 1, next choice as 2, and so on.				
Martin Scorsese, *The Irishman*	1	2	3	4	5
Todd Phillips, *Joker*	1	2	3	4	5
Sam Mendes, *1917*	1	2	3	4	5
Quentin Tarantino, *Once Upon a Time in Hollywood*	1	2	3	4	5
Bong Joon-ho, *Parasite*	1	2	3	4	5

As you decide on the voting methods that will be used in your new democracy, budget must be a consideration. You might consider a particular type of ranked voting called **ranked-choice voting (RCV)**, which simulates a series of runoff elections without the usual time and expense involved when voters must repeatedly return to the polls, like we did in Example 4.

The method of ranked-choice voting (RCV), also called **instant runoff voting (IRV)**, is a version of the Hare Method, using *preference ranking* so that, if no single candidate receives a majority, the least popular selections can be eliminated and the results can be recounted, without the need for more elections.

CHECKPOINT

Ranked voting can be confused with ranked-choice voting, but ranked voting is a more general category which includes ranked-choice voting and several other voting methods.

As we explore examples of ranked voting, we will summarize the voters' preference rankings using a table in which the top row shows the number of ballots that ranked the options in the same order. Let's practice interpreting the information in this type of table.

Example 5 **Interpreting the Sample Preference Summary**

Refer to the table below containing voters' preference rankings to answer the following questions.

Number of Ballots	100	200	150	75
Option A	1	4	3	4
Option B	2	3	4	2
Option C	4	2	1	1
Option D	3	1	2	3

1. How many voters ranked the options in the following order: Option A in fourth place, Option B in second place, Option C in first place, and Option D in third place?
2. How many ballots in total were collected?
3. How many voters indicated that Option C was their first choice?

Solution

1. The column farthest to the right displays this ordering. The top entry in this column is 75; so, there were 75 voters who ranked the options in this way.
2. The sum of the top row gives the total number of ballots collected: $100 + 200 + 150 + 75 = 525$. So, there were 525 ballots collected.
3. In the Option C row, there are two entries of 1 which indicate a first choice for that option. These occur in the last two columns. The sum of the top entries in these columns is $150 + 75 = 225$. So, 225 voters indicated Option C as their first choice.

Now that we've covered how to read a summary of preference rankings, let's practice using the ranked-choice method to determine the winner of an election. Recall that ranked-choice voting is still the Hare Method where the candidate with the very least number of votes is eliminated each round until a majority is attained. The difference here is that the voters have completed a ranked ballot, so they don't have to visit the polls multiple times. Here are the steps for ranked-choice voting.

Steps to Determine Winner by Ranked-Choice Voting

To determine the winner when ranked-choice voting occurs, we take these three steps:

Step 1: Determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election and we are done! Otherwise, eliminate the candidate(s) with the fewest votes and complete Step 3.

Step 3: Reallocate the votes to the remaining candidates, and repeat Step 2.

CHECKPOINT

Be methodical to avoid arithmetic errors. Make sure that each time you count the number of first place votes they sum to the number of ballots.

VIDEO

How Does Ranked-Choice Voting Work?

Example 6 **Most Popular Color in Kindergarten**

Let's review the kindergarten class color preferences again, and this time determine which color would be selected based on these results using the ranked-choice method.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Solution

Step 1: Determine whether any candidate received a majority. There were 21 ballots. Fifty percent of 21 is $0.50(21) = 10.5$. A majority is 11.

Step 2: Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Red: 0
- Blue: $4 + 4 = 8$
- Green: 0
- Yellow: 0
- Purple: 7

- Pink: 6

  Notice that $8 + 7 + 6 = 21$, which is the total number of ballots. Confirming this helps to catch any arithmetic or counting errors. No candidate has a majority with 11 or more votes. We must eliminate red, green, and yellow which had the fewest votes with 0 each. The remaining votes that must be counted for Round 2 are given in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	4
Purple	4	3	4	1
Pink	3	1	3	6

Step 3: Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1, 2, and 3 in such a way that the order of preference is retained as seen in the table below:

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1
Pink	2	1	2	3

Step 4: Repeat the process from Step 2. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 4 = 8$

- Purple: 7
- Pink: 6

  Confirm that $8 + 7 + 6 = 21$. Great! No candidate has 11 or more votes. We must eliminate pink which had the fewest votes with 6. The remaining votes that must be counted for Round 2 are shown in the table below.

Number of Ballots	4	6	4	7
Blue	1	2	1	2
Purple	3	3	3	1

Step 5: Repeat the process from Step 3. Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1 and 2;

Number of Ballots	4	6	4	7
Blue	1	1	1	2
Purple	2	2	2	1

Step 6: Repeat the process from Step 2 one last time. Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

- Blue: $4 + 6 + 4 = 14$
- Purple: 7

Blue has a majority and wins the election!

VIDEO

Determine Winner of Election by Ranked-Choice Method (aka Instant Runoff)

Borda Count Voting

Ranked-choice voting is one type of ranked voting that simulates multiple runoffs based on ranked ballots. Another type of ranked voting is the **Borda count method**, which uses ranked ballots that award candidates points corresponding to the number of candidates ranked lower on each ballot.

To understand how this works, let's review the favorite colors of our kindergarten class from the table below. Let's focus on the votes represented by the first column of the preference summary.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Each student had six options. This first column tells us that four students ranked blue as their first choice, red as their second choice, pink as their third choice, purple as their fourth choice, yellow as their fifth choice, and green as their sixth choice. Blue was ranked higher than $6 - 1 = 5$ other colors. For each of the four students who completed their ballot in this way, blue would receive five points. Since there were four ballots with this ordering, blue would receive $5(4) = 20$ points from the first column. To determine the total points for each candidate, we have to find the sum of the points they received in each column.

To determine the winner of a contest using the Borda count method, we must compare total number of points earned by each candidate. The candidate with the most points is the winner. Each row of the preference summary corresponds to a single candidate. To find the number of points received by a particular candidate in the preference summary, or their **Borda score**, we will need to focus on the row in which that candidate appears.

Before we practice determining the winner of a Borda count election, let's examine how to find the Borda score for a single candidate.

Example 7 **Most Popular Color in Kindergarten Revisited**

Let's review the ballots from the kindergarten class again, as shown in the table below. This time, let's determine the Borda score received by the color purple.

Number of Ballots	4	6	4	7
Red	2	6	2	5
Blue	1	2	1	4
Green	6	5	5	2
Yellow	5	4	6	3
Purple	4	3	4	1
Pink	3	1	3	6

Solution

Step 1: Find the number of points received by the candidate in each column.

Column 1: $4 \times (6 - 4) = 4 \times 2 = 8$

Column 2: $6 \times (6 - 3) = 6 \times 3 = 18$

Column 3: $4 \times (6 - 4) = 4 \times 2 = 8$

Column 4: $7 \times (6 - 1) = 7 \times 5 = 35$

Step 2: Find the sum of the points received in each column. This is the total number of points received by this candidate: $8 + 18 + 8 + 35 = 69$

This process can also be combined into one step as shown here.

$$\begin{aligned} &4 \times (6 - 4) + 6 \times (6 - 3) + 4 \times (6 - 4) + 7 \times (6 - 1) \\ &= 4 \times 2 + 6 \times 3 + 4 \times 2 + 7 \times 5 \\ &= 8 + 18 + 8 + 35 \\ &= 69 \end{aligned}$$

Purple received 69 points in this election.

CHECKPOINT

When calculating a Borda score in one step, be careful to use the correct order of operations.

Perform the subtraction inside each pair of parentheses first, then perform each multiplication, and then perform each addition.

Now let's determine the winner of an election by comparing the Borda scores for each of the candidates.

Example 8 **Determine the Winner by Two Ranked Voting Methods**

Use the table below, which displays a sample preference summary, to answer the questions that follow.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.

Solution

1. **Step 1:** Determine the number of votes needed to achieve a majority. The number of ballots is $95 + 90 + 110 + 115 = 410$. Fifty percent of 410 is $0.50(410) = 205$. So, 206 votes or more is a majority.

Step 2: Count the number of first place votes for each candidate.

- Option A has $110 + 115 = 225$
- Option B has 0
- Option C has 90
- Option D has 95

Since Option A has a majority, Option A is the winner by the ranked-choice method.

1. The Borda scores would be:
   - Option A: $95(4-4) + 90(4-4) + 110(4-1) + 115(4-1) = 675$
- Option B: $95(4-2) + 90(4-2) + 110(4-2) + 115(4-2) = 820$
- Option C: $95(4-3) + 90(4-1) + 110(4-3) + 115(4-4) = 475$
- Option D: $95(4-1) + 90(4-3) + 110(4-4) + 115(4-3) = 490$

Since Option B has a Borda score of 820 points, Option B is the winner by the Borda count method.

The Borda count method may seem too complicated to even consider using for Imaginaria, but each voting method has its own pros and cons. The Borda count method, for example, favors compromise candidates over divisive candidates. A **compromise candidate** is not the first choice of most of the voters, but is more acceptable to the population as a whole than the other candidates. A **divisive candidate** is simultaneously the first choice of a large portion of the voters and the last choice of another large portion of the voters.

In Example 8, Candidate A was ranked first by 225 voters, but was ranked last by 185 voters. No voters ranked Candidate A as second or third. It appears that, although Candidate A had the majority of first place votes, there was a significant minority who strongly disliked them. Candidate A was a divisive candidate. Candidate B, on the other hand, was the second choice of every voter, making Candidate B a good compromise. The Borda count method chose Candidate B, a compromise candidate, that was more acceptable to the population as a whole. This scenario is cited by both opponents and proponents of the Borda count method.

> **VIDEO**
>
> Determine Winner of Election by Borda Count Method

Pairwise Comparison and Condorcet Voting

We have discussed two kinds of ranked voting methods so far: ranked-choice and Borda count. A third type of ranked voting is the **pairwise comparison method**, in which the candidates receive a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie. If one candidate earns more points than the others, then that candidate wins. This method is one of several **Condorcet voting methods**, which are methods in which candidates are ranked and then compared pairwise to each other, a candidate having to beat all others in order to win. These methods vary in the way candidates are scored, and there is not always a clear winner. A candidate who wins each possible pairing is known as a **Condorcet candidate**. These terms are named after the Marquis de Condorcet, a French philosopher and mathematician who preferred the pairwise comparison method to the Hare method and made public arguments in its favor.

> **PEOPLE IN MATHEMATICS**
>
> *Marquis de Condorcet*
>
> Condorcet voting methods are named for the Marquis de Condorcet, a French philosopher and mathematician known for, among other accomplishments, writing "*Sur l'admission des femmes au droit de Cité*" ("On the Admission of Women to the Rights of Citizenship"), in 1789, the first published essay on the political rights of women.
>
> For more details visit this Web site.

If you include a Condorcet voting method in the constitution of Imaginaria, the election supervisors may want to use a pairwise comparison matrix like the one in . It's a tool used to list the number of wins associated with each pairing of two candidates. Each candidate will receive a point for each win and a half a point for each tie. Each pairing is listed twice, once for the number of wins of a candidate over a particular challenger and once for the number of wins of the challenger over that candidate.

Runner \ Opponent	A	B	C
A wins	--	A over B	A over C
B wins	B over A	--	B over C
C wins	C over A	C over B	--

Figure 3: Pairwise Comparison Matrix for Three Candidates

Steps to Determine a Winner by Pairwise Comparison Method Using a Matrix

To determine the winner when the pairwise comparison method is used, we take these three steps:

Step 1: On the matrix, indicate a losing matchup by crossing out a box, $\boxed{\times}$, and tie match ups by drawing a slash through the box, $\boxed{\backslash}$.

Step 2: Award each candidate 1 point for a win, half a point for a tie, and 0 points for a loss.

Step 3: Identify the winner, which is the candidate with the most points.

VIDEO

Determine Winner of Election by Using the Pairwise Comparison Method

Before you decide on the pairwise comparison method for Imaginaria, review what's involved in constructing a pairwise comparison matrix from a summary of ranked ballots. Then we can use the matrix to determine the winner of the election. Does the winner using the Borda method still win?

Example 9 **Construct and Use a Pairwise Comparison Matrix**

Consider the summary of ranked ballots shown in the table below. Determine the winner of an election using the pairwise comparison method.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

1. Construct a pairwise comparison matrix for the sample summary of ranked ballots in the table above.

2. Use the pairwise comparison method to determine a winner.
3. Recall that in Example 8, Candidate A won by the ranked-ballot method, and Candidate B won by the Hare method. Did the same candidate win using the pairwise comparison method?
4. Is the winner a Condorcet candidate?

Solution

There are four candidates on the ballots. We will need a row and a column for each candidate in addition to the headings, so we will draw a five by five matrix.

Opponent / Runner	Option A	Option B	Option C	Option D
Option A wins	--	A over B	A over C	A over D
Option B wins	B over A	--	B over C	B over D
Option C wins	C over A	C over B	--	C over D
Option D wins	D over A	D over B	D over C	--

Figure 4: Pairwise Comparison Matrix for Four Candidates

1. **Step 1:** Refer to to determine the values that belong in each cell.

- A over B: A is preferred to B in columns 3 and 4. So, A scores $110 + 115 = 225$ points.
- A over C: A is preferred to C in columns 3 and 4. So, A scores 225 points again.
- A over D: Similarly, A scores 225 points.
- B over A: B is preferred to A in columns 1 and 2. So, B scores $95 + 90 = 185$ points.

Step 2: Continuing in this way, we complete the pairwise comparison matrix, as shown.

Opponent / Runner	Option A	Option B	Option C	Option D
Option A wins	--	AB 225	AC 225	AD 225
Option B wins	BA 185	--	BC 320	BD 315
Option C wins	CA 185	CB 90	--	CD 200
Option D wins	DA 185	DB 95	DC 210	--

Figure 5: Pairwise Comparison Matrix for Four Candidates with Vote Counts

1. **Step 1:** Losing pairings are crossed off with an $\boxed{\times}$. In the event of a tie, we will draw a slash, $\boxed{\backslash}$.

Step 2: Determine the number of points for each candidate by analyzing their row of wins. Each win is 1 point, each loss, $\boxed{\times}$, is 0 points, and each tie, $\boxed{\backslash}$, is half a point. Construct an additional column for each candidate's points.

Opponent / Runner	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	~~BA 185~~	--	BC 320	BD 315	2
Option C wins	~~CA 185~~	~~CB 90~~	--	~~CD 200~~	0
Option D wins	~~DA 185~~	~~DB 95~~	DC 210	--	1

Figure 6: Pairwise Comparison Matrix for Four Candidates with Pairwise Winners and Points Column Added

Step 3: The winner by the pairwise comparison method is Option A with 3 points.

1. Option A was not the winner by the Hare method.
2. The winner, Option A, is a Condorcet candidate because Option A won each pairwise comparison.

CHECKPOINT

Notice that the pairwise vote totals are not used to determine the points. Vote totals are only used to determine a win or a loss. Avoid the common error of adding the values in each row to get the points.

Three Key Questions

Before you decide if you want to use the pairwise comparison method for Imaginarian elections, let's consider three questions that might affect your decision.

1. Is there always a winner?
2. If there is a winner, is the winner always a Condorcet candidate?
3. If there is a Condorcet candidate, does that candidate always win?

   Let's think about why these questions might be important to you if you chose the pairwise comparison method. First, if no candidate meets the criteria to win an election, you will need a backup plan such as a runoff election. Second, if the winner is not a Condorcet candidate, then there is at least one candidate who beat the winner in a pairwise matchup and the supporters of that candidate might question the validity of the election. Finally, if there is a Condorcet candidate who beat every other candidate in a pairwise matchup, it is reasonable to conclude that it would be unfair for anyone else to win. The rest of the examples in this section should illustrate these key concepts.

Example 10 **Rock, Paper, Scissors by Pairwise Comparison**

Suppose that three people are playing the game Rock, Paper, Scissors. On the count of three, each person shows a hand signal for rock, paper, or scissors. Each hand signal beats another hand signal. The group keeps having a tie because Person A always picks rock, Person B always picks paper which beats rock, and Person C always picks scissors which beats paper, and is beaten by rock! This leads to a disagreement about which choice is best. They decide to use the pairwise comparison method determine the winner. Their preference rankings are given in the following table.

Voters	A	B	C
Rock (R)	1	3	2
Paper (P)	2	1	3
Scissors (S)	3	2	1

Solution

Construct the comparison matrix:

Opponent / Runner	Rock (R)	Paper (P)	Scissors (S)	Points
Rock wins	--	RP 2	RS 1	1
Paper wins	PR 1	--	PS 2	1
Scissors wins	SR 2	SP 1	--	1

Figure 7: Rock, Paper, Scissors Pairwise Comparison Matrix

There is a tie! There is no winner.

Example 10 illustrates the answer to the first key question. The pairwise comparison method does not always result in a winner. For example, much like the game of Rock, Paper, Scissors, it is possible for a cyclic pattern to emerge in which each candidate beats the next until the last candidate who beats the first.

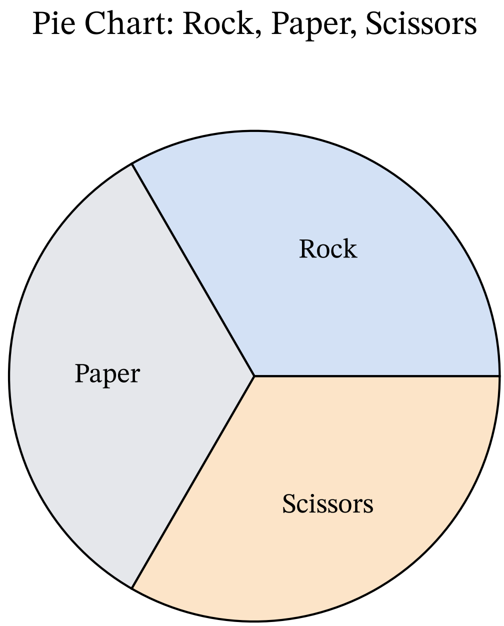

Figure 8: Rock, Paper, Scissors Cyclic Outcome

Now, you have the answer to the second key question. The pairwise comparison matrix in YOUR TURN 11.10 is an example of a scenario where a winner is not a Condorcet candidate.

The answer to the third question is not as clear. If there is a Condorcet candidate, does that candidate win? So far, we have not come across a contradictory example where the Condorcet candidate didn't win, but we cannot know with certainty that it is not possible by looking at examples. Instead, we will need to use some reasoning. Let's review some particular cases of elections with a certain number of candidates, and then we will try to generalize the scenario to an election with *n* candidates.

Example 11 **Does the Condorcet Candidate Win?**

1. Suppose there is an election with five candidates—A, B, C, D, and E—and that Candidate C is a Condorcet candidate. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

Solution

1. In any pairwise election with five candidates, each candidate must compete against four other candidates. It follows that the most points a single candidate can win is four points, which would occur if the candidate won every matchup. As a Condorcet candidate, Candidate C won all the pairwise matchups against Candidates A, B, D, and E, earning four points.

2. The rest of the candidates lost to Candidate C. The most points a particular candidate could win if they won matchups with each of the other three candidates is three points.
3. Since Candidate C has four points and the rest of the candidates have three points or less, Candidate C is the winner. Therefore, it is not possible for Candidate C to tie or lose.

Let's consider a general case where there are n candidates. One of the candidates is a Condorcet candidate. Since the Condorcet candidate wins all matchups, the Condorcet candidate wins $n-1$ points. Since each of the other candidates lost to the Condorcet candidate, the most a single candidate could win is $n-2$. Since the Condorcet candidate won $n-1$ points and each other candidate won $n-2$ points or fewer, the Condorcet candidate is the winner. You have your answer to the third key question! If there is a Condorcet candidate, that candidate is always the winner.

Approval Voting

The last type of voting system you will consider for your budding democracy is an **approval voting system**. In this system, each voter may approve any number of candidates without rank or preference for one over another (among the approved candidates), and the candidate approved by the most voters wins. This voting system has aspects in common with plurality voting and Condorcet voting methods, but it has characteristics that distinguish it from both. An **approval voting ballot** lists the candidates and provides the option to approve or not approve each candidate.

The term "approval voting" was not used until the 1970s Brams, Steven J.; Fishburn, Peter C. (2007), *Approval Voting*, Springer-Verlag, p. xv, ISBN 978-0-387-49895-9, although its use has been documented as early as the 13th century (Brams, Steven J. (April 1, 2006). *The Normative Turn in Public Choice* (PDF) (Speech). Presidential Address to Public Choice Society. New Orleans, Louisiana.) Approval voting has the appeal of being simpler than ranked voting methods. It also allows an individual voter to support more than one candidate equally. This has appeal for those who do not want a split vote among a few mainstream candidates to lead to the election of a fringe candidate. It also has appeal for those who want an underdog to have a chance of success because voters will not worry about wasting their vote on a candidate who is not believed likely to win.

Example 12 **Rock, Paper, Scissors, Lizard, Spock**

Suppose that Person A/B/C were just about to give up on their game of Rock, Paper, Scissors when they were joined by Person D who reminded them that their updated version, Rock, Paper, Scissors, Lizard, Spock was a far superior game with the added rules that Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and Scissors decapitates Lizard.

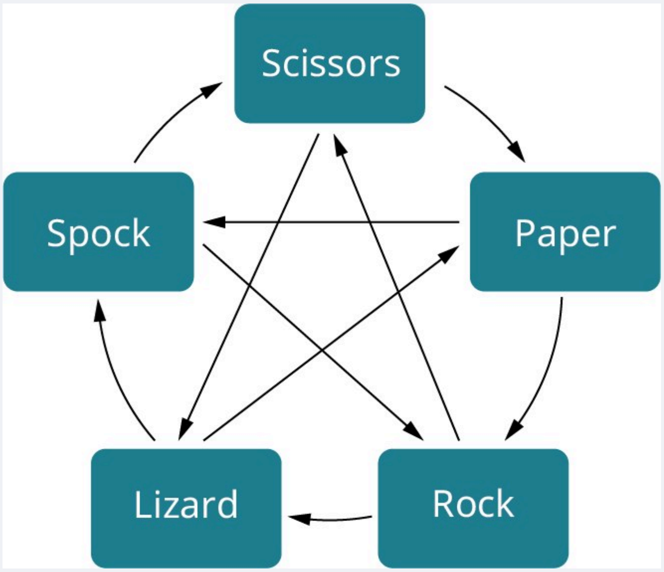

Figure 9: Rock, Paper, Scissors, Lizard, Spock Dominance

Person D encourages their friends to hold a new election. This time, for the sake of simplicity, the group decides to use approval voting to determine the best move in the game. The summary of approval ballots for Rock, Paper, Scissors, Lizard, Spock is given in the table below.

Voters	A	B	C	D
Rock	Yes	No	No	No
Paper	No	Yes	No	No
Scissors	No	No	Yes	No
Lizard	No	No	No	Yes
Spock	Yes	Yes	Yes	Yes

Solution

Count the number of approval votes for each candidate by counting the number of “Yes” votes in each row of the table.

- Rock: 1
- Paper: 1
- Scissors: 1
- Lizard: 1
- Spock: 4

  Spock is the winning candidate, approved by four voters!

Example 13 **The Chionilis Family Is Hungry Again!**

The eight members of the Chionilis family—Annette, Rene, Seema, Titus, Galen, Elena, Max and Demitri—have another decision to make. Approval voting worked out nicely the last time. They are going to use it again, but this time, Annette, Rene, Seema, and Galen are feeling a little indecisive. They can't narrow their choice down to two. They will approve their three top choices, but the other family members will only approve two. These choices are reflected in the following table. Determine the restaurant that will be chosen.

Options	A	R	S	T	G	E	M	D
Rainbow China	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
Dough Boys Pizza	Yes	Yes	Yes	Yes	No	Yes	No	Yes
Taco City	Yes	No	Yes	No	Yes	Yes	No	No
Caribbean Flavor	No	Yes	No	No	Yes	No	Yes	No

Solution

Count the number of approval votes for each restaurant by counting the number of "Yes" votes in each row.

- Rainbow China: 7
- Dough Boys Pizza: 6
- Taco City: 4
- Caribbean Flavor: 3

  This time, Rainbow China won!

Compare and Contrast Voting Methods to Identify Flaws

Wow! We have covered a lot of options for the voting methods. Now, you need to decide which one is best for Imaginaria. Imaginarians might consider characteristics of certain voting systems desirable and others undesirable. In some cases, voters may consider these undesirable traits to be flaws in a voting system that are significant enough to motivate them to reject that system. If you are feeling a bit overwhelmed by this decision, maybe it would help to read about the experiences of others who have faced similar questions.

Consider the 2000 U.S. presidential election in which Green Party candidate Ralph Nader and Reform Party candidate Pat Buchanan were on the ballet running against the mainstream candidates, Democrat Al Gore and Republican George W. Bush. The voting results for Florida are given in .

Candidate	Party	Votes	Percentage
(G) George W. Bush	Republican	2,912,790	48.85%
(A) Al Gore	Democrat	2,912,253	48.84%
(R) Ralph Nader	Green	97,488	1.63%
(P) Pat Buchanan	Reform	17,484	0.29%
(H) Harry Brown	Libertarian	16,415	0.28%
(O) 7 Other Candidates	Other	6,680	0.11%
	Total	5,963,110	

In more than one state, Buchanan was able to split the Republican vote enough to allow Gore to win that state. Nader split the Democrat vote in Florida and New Hampshire by enough votes to prevent Gore from winning those states. Had Gore won either state, he would have had enough electoral votes to win the election. Instead, Bush won. This is an example of a flaw in the plurality system of voting: the spoiler.

A **spoiler** is a less popular candidate who takes votes from a more popular candidate with similar positions, swinging the race to another candidate with vastly different views that they would not support. This encourages voters not to vote for the candidate that they perceive to be the best, but instead for the candidate they can live with who they perceive to have a better chance of winning. Some voters may prefer a method such as approval voting, which does not have this trait in common with plurality voting.

Example 14 **The Spoiler Controversy**

Because the vote counts for George W. Bush and Al Gore differed by only 537 votes, many Democrats blamed Ralph Nader and the Green Party for their loss. Let's consider how the election results might have differed if the approval voting method had been used.

Use and the following assumptions to extrapolate the results of an approval method election:

- 100 percent of Pat Buchanan supporters would approve George W. Bush.
- 100 percent of Ralph Nader supporters would approve Al Gore.
- 72 percent of Libertarians would approve George W. Bush.
- 28 percent of Libertarians would approve Al Gore (as was roughly the known percentage at the time according to the Cato Institute).
- 50 percent of the supporters of other candidates would approve George Bush while 50 percent would approve Al Gore.

Solution

Step 1: Create a summary of approval ballots based on the given assumptions. For the Libertarian candidate, 72 percent of 16,415 of the votes is $0.72(16,415) = 11,819$ and

28 percent is $0.28(16,415) = 4,596$. For the other candidates, 50 percent of the votes is $0.50(6,680) = 3,340$.

Number of Votes	2,912,790 (G)	2,912,253 (A)	97,488 (R)	17,484 (B)	11,819 (72% H)	4,596 (28% H)	3,340 (50% O)	3,340 (50% O)
(G) George W. Bush	Yes	No	No	Yes	Yes	No	No	Yes
(A) Al Gore	No	Yes	Yes	No	No	Yes	Yes	No
(R) Ralph Nader	No	No	Yes	No	No	No	No	No
(P) Pat Buchanan	No	No	No	Yes	No	No	No	No
(H) Harry Brown	No	No	No	No	Yes	Yes	No	No
(O) 7 Other Candidates	No	No	No	No	No	No	Yes	Yes

Step 2: Count the number of approval votes for each candidate.

- George W. Bush: $2,912,790 + 17,484 + 11,819 + 3340 = 2,945,433$
- Al Gore: $2,912,253 + 97,488 + 4,596 + 3,340 = 3,017,677$
- Ralph Nader: $97,488$
- Pat Buchanan: $17,484$
- Harry Brown: $11,819 + 4,596 = 16,415$
- Other Candidates: $6,680$

  In this scenario, Al Gore is the winner.

The results in Example 14 and Your Turn 11.14 highlight one of the characteristics of approval voting. Ralph Nader moved up from a distant third place finish to a close second place finish when Al Gore's supporters approved him on their ballots. In this way, fringe candidates have a better chance of winning, which some voters consider a flaw but others consider a benefit.

Another aspect of approval voting systems that is a concern to many voters is that candidates in approval elections might encourage their loyal supporters to approve them and only them to avoid giving support to any other candidate. If this occurred, the election in effect becomes a traditional plurality election. This is a flaw that cannot occur in an instant runoff system since all candidates are ranked.

Example 15 **Three Habitable Planets**

In the future, humans have explored distant solar systems and found three habitable planets which could be colonized. Since it will take all available resources to colonize one planet, humans must agree on the planet. Planet A has the most comfortable climate and most plentiful resources, but it is the farthest from Earth making travel to the planet a challenge. Planet B is half the distance but will require more resources to make comfortable. Planet C is the least suitable of the three and terraforming will be required, but it is close enough to make travel between Earth and Planet C possible on a more regular basis. The table below provides the voter preferences for the colonization of each planet.

Percentage of Voters	45%	15%	40%
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

If the entire population were able to vote, determine the winning planet using each of the methods listed below.

1. Plurality
2. Ranked-choice method
3. Borda count

Solution

1. The plurality method only considers the top choice of each voter. By this system, Planet A has 45 percent of the vote, Planet B has 15 percent of the vote, and Planet C has 40 percent of the vote. Planet A wins.
2. Using either instant runoff or a two-round system, Planet B with only 15 percent of the vote will be eliminated in the first round. In Round 2, the 15 percent that voted for Planet B would vote for their second choice, Planet C. This leaves Planet A with 45 percent and Planet C with 55 percent. Planet C has a majority and wins the election.
3. To find the Borda score for each candidate, imagine there are exactly 100 voters. Then the summary of ranked ballots looks like:

Out of 100 Voters	45	15	40
Planet A	1	3	3
Planet B	2	1	2
Planet C	3	2	1

The Borda score for each candidate is as follows:
Planet A: $(3-1)(45)+(3-3)(15)+(3-3)(40)=90$
Planet B: $(3-2)(45)+(3-1)(15)+(3-2)(40)=115$
Planet C: $(3-3)(45)+(3-2)(15)+(3-1)(40)=95$
Planet B wins.

The election in Example 15 involves a scenario in which there are two extreme candidates, Planet A and Planet B, and a moderate candidate, Planet C. The supporters of the extreme candidates prefer the moderate candidate to the other extremist ones. This makes Planet C a compromise candidate. In this case, both the plurality method and ranked-choice voting resulted in the election of one of the extreme candidates, but the Borda count method elected the compromise candidate in this scenario. Depending on a person's perspective, this may be perceived as a flaw in either ranked-choice and plurality systems, or the Borda count method.

In Fairness in Voting Methods, we will analyze the fairness of each voting system in greater detail using objective measures of fairness.

TECH CHECK

Voting Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to determine the winner of an election by various voting methods. In some cases, this work has already been done and posted online. As you practice applying the various voting methods that could be used in Imaginaria, quick Internet search will lead to sites such as Ms. Hearn Math with free specialty calculators.

These sites can be a great way to check your results!

Key Terms

- majority
- plurality
- runoff election
- runoff voting system
- two-round system
- Hare Method

- preference ranking
- ranked ballot
- ranked-choice voting (RCV)
- instant runoff voting (IRV)
- Borda count method
- Borda score
- Compromise candidate
- divisive candidate
- pairwise comparison method
- Condorcet voting methods
- Condorcet candidate
- approval voting system
- approval voting ballot
- spoiler

Key Concepts

- In plurality voting, the candidate with the most votes wins.
- When a voting method does not result in a winner, runoff voting can be used to do so.
- Ranked-choice voting, also known as instant runoff voting, is one type of ranked voting system.
- The Borda count method is a type of ranked voting system in which each candidate is given a Borda score based on the number of candidates ranked lower than them on each ballot.
- When pairwise comparison is used, the winner will be the Condorcet candidate if one exists.
- Approval voting allows voters to give equally weighted votes to multiple candidates.
- When a voter finds a characteristic of a particular voting method unappealing, they may consider that characteristic a flaw in the voting method and look for an alternative method that does not have that characteristic.

Videos

- How Does Ranked-Choice Voting Work?
- Determine Winner of Election by Ranked-Choice Method (aka Instant Runoff)
- Determine Winner of Election by Borda Count Method
- Determine Winner of Election by Pairwise Comparison Method

11.2 Fairness in Voting Methods

Figure 10: Citizens strive to ensure their voting system is fair.

Learning Objectives

After completing this section, you should be able to:

1. Compare and contrast fairness of voting using majority criterion.
2. Compare and contrast fairness of voting using head-to-head criterion.
3. Compare and contrast fairness of voting using monotonicity criterion.
4. Compare and contrast fairness of voting using irrelevant alternatives criterion.
5. Apply Arrow's Impossibility Theorem when evaluating voting fairness.

Now that we've covered a variety of voting methods and discussed their differences and similarities, you might be leaning toward one method over another. You will need to convince the other founders of Imaginaria that your preference will be the best for the country. Before your collaborators approve the inclusion of a voting method in the constitution, they will want to know that the voting method is a fair method. In this section, we will formally define the characteristics of a fair system. We will analyze each voting previously discussed to determine which characteristics of fairness they have, and which they do not. In order to guarantee one ideal, we must often sacrifice others.

The Majority Criterion

One of the most fundamental concepts in voting is the idea that most voters should be in favor of a candidate for a candidate to win, and that a candidate should not win without majority support. This concept is known as the **majority criterion**.

With respect to the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—we will explore two important questions:

1. Which of these voting systems satisfy the majority criterion and which do not?
2. Is it always "fair" for a voting system to satisfy the majority criterion?

   Keep in mind that this criterion only applies when one of the candidates has a majority. So, the examples we will analyze will be based on scenarios in which a single candidate has more than 50 percent of the vote.

Example 1 **Roommates Choose Fast Food**

It's final exams week and seven college students are hungry. They must get food, but from which drive thru? Their preferences are listed in the table below. The majority have listed McDonald's as their top choice. Let's calculate what the results of the election will be using various voting methods.

Voters	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the plurality voting method?
2. Which restaurant is the winner using the ranked-choice voting method?
3. Does the majority criterion apply? If so, for which of voting method(s), if any, did the majority criterion fail?

Solution

1. For plurality voting, we only need to count the first-place votes for each candidate. In this case, McDonald's has four first place votes, which is a majority and wins the election automatically.
2. For ranked-choice voting, McDonald's also wins because it has a majority at the end of Round 1.
3. The majority criterion does apply because one candidate had a majority of the first-place votes. The majority criterion did not fail by either method, because in each case the majority candidate won.

From Example 1, it appears that the plurality and ranked-choice voting methods satisfy the majority criterion. In general, the majority candidate always wins in a plurality election because the candidate that has more than half of the votes has more votes than any other candidate. The same is true for ranked-choice voting; and there will never be a need for a second round when there is a majority candidate. Let's examine how some of the other voting methods stand up to the majority criterion.

Example 2 **Roommates Choose Fast Food**

Those seven college students are hungry again! Their preferences haven't changed, as shown below. Let's calculate if the results change when we use different voting methods.

VOTERS	A	B	C	D	E	F	G
(M) McDonald's	1	1	1	1	5	5	5
(B) Burger King	2	2	2	2	2	2	2
(T) Taco Bell	4	4	3	4	3	3	1
(O) Pollo Tropical	3	5	4	3	4	1	3
(I) Pizza Hut	5	3	5	5	1	4	4

1. Which restaurant is the winner using the pairwise comparison voting method?
2. Which restaurant is the winner using the Borda count voting method?
3. Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

Solution

1. For pairwise comparison, notice that McDonald's is a Condorcet candidate because it wins every pairwise comparison. So, McDonald's is the winner.
2. For the Borda count, we must calculate the Borda score for each candidate: McDonald's is 16, Burger King is 21, Taco Bell is 13, Pollo Tropical is 12, Pizza Hut is 8. The winner is Burger King!
3. Yes, the majority criterion applies because McDonald's has the majority of first place votes. The majority criterion only fails using the Borda method.

Example 2 demonstrates a concept that we also saw in Borda Count Voting—the Borda method frequently favors the compromise candidate over the divisive candidate. This can happen even when the divisive candidate has a majority, as it did in this example. Although a majority of the voters were in favor of McDonald's, a significant minority was strongly opposed to McDonald's, ranking it last. Since the Borda score includes all rankings, this strong opposition has an impact on the outcome of the election.

Pairwise comparison will always satisfy the majority criterion because the candidate with the majority of first-place votes wins each pairwise matchup. While it is possible for the majority candidate to win by the Borda count method, it is not guaranteed. So, the Borda method fails the majority criterion. A summary of each voting method as it relates to the majority criterion is found in the following table.

Voting Method	Majority Criterion
Plurality	Satisfies
Ranked-choice	Satisfies
Pairwise comparison	Satisfies
Borda count	Violates

If you prefer the Borda method, you might argue that its failure to satisfy the majority criterion is actually one of its strengths. As we saw in Example 1 and Example 2, the majority have the power to vote for their own benefit at the expense of the minority. While four students were very enthusiastic about McDonald's, three students were strongly opposed to McDonald's. It is reasonable to say that the better option would be Burger King, the compromise candidate, which everyone ranked highly and no one strongly opposed. The Borda method is designed to favor a candidate that is acceptable to the population as a whole. In this way, the Borda method avoids a downfall of strict majority rule known as the **tyranny of the majority**, which occurs when a minority of a population is treated unfairly because their situation is different from the situation of the majority.

The people of Imaginaria should know that the power of the majority to vote their will has serious implications for other groups. For example, according to the UCLA School of Law Williams Institute, the LGBTQ+ community in the United States makes up approximately 4.5 percent of the population. When elections occur that include issues that affect the LGBTQ+ community, members of the LGBTQ+ community depend on the 95.5 percent of the population who do not identify as LGBTQ+ to consider their perspectives when voting on issues such as same-sex marriage, the use of public restrooms by transgender people, and adoption by same-sex couples.

PEOPLE IN MATHEMATICS

Robert Dahl

In his book, *Democracy and Its Critics*, Robert Dahl wrote, “If a majority is not entitled to do so, then it is thereby deprived of its rights; but if a majority is entitled to do so, then it can deprive the minority of its rights.” Dahl was a renowned political theorist, but he is also considered to be a mathematician since his work utilizes ideas from an area of mathematics known as Game Theory (Mathematics Genealogy Project, NDSU Department of Mathematics with the American Mathematical Society).

WHO KNEW?

Three Branches of Government

Concerns about the consequences of majority rule are not new. In 1788, John Adams warned of the consequences of majority rule and he argued for three branches of government as a way to temper them. In the early 1800s, a young French aristocrat named Alexis de Tocqueville toured the United States and wrote *Democracy in America*, which focused on the impact of democracy on political and civil societies. He observed, even then, the dominance of the white majority over the Indigenous people and enslaved people, which was perpetuated by majority rule.

VIDEO

Separation of Powers and Checks and Balances

Head-to-Head Criterion

Another fairness criterion you must consider as you select a voting method for Imaginaria is the **Condorcet criterion**, also known as the head-to-head criterion. An election method satisfies the Condorcet criterion provided that the Condorcet candidate wins the election whenever a Condorcet candidate exists. A **Condorcet method** is any voting method that satisfies the Condorcet criterion.

Recall from Three Key Questions that not every election has a Condorcet candidate; the Condorcet criterion will not apply to every election. Also recall that a Condorcet candidate cannot lose an election by pairwise comparison. So, the pairwise comparison voting method is said to satisfy the Condorcet criterion.

Example 3 **Spending Tax Refund**

A survey asked a random sample of 100 people in the United States to rank their priorities for spending their tax refund. The options were (V) go on vacation, (S) put into savings, (D) pay off debt, or (T) other. The pairwise comparison matrix for the results is in . Determine whether the Condorcet criterion applies.

Opponent Runner	V	S	D	T
V wins	--	~~VS 33~~	~~VD 37~~	VT 100
S wins	SV 67	--	~~SD 36~~	ST 100
D wins	DV 63	DS 64	--	DT 96
T wins	~~TV 0~~	~~TS 0~~	~~TD 4~~	--

Figure 11: Pairwise Comparison Matrix for Tax Refund Spending

Solution

The Condorcet criterion only applies when there is a Condorcet candidate. “Pay off debt” (D) is a Condorcet candidate because D wins every matchup. Yes, the Condorcet criterion applies to this election.

Example 4 **Spending Tax Refund**

Let's return to the survey about tax refund spending from Example 3. We know that the Condorcet criterion applies because Option D, "Pay off debt," is a Condorcet candidate, which wins every pairwise match up.

Use the information in the ballot summary from the table below to find the winner and determine whether the Condorcet criterion is satisfied in this election when each of the following voting methods are used.

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	4
Other (T)	4	4	4	3

1. Plurality
2. Ranked-choice voting
3. Borda count

Solution

1. V wins 33 first place votes; S, 36; D, 31; and T, 0. So candidate S, "Put into savings," has a plurality and wins. Since the Condorcet candidate D didn't win, the Condorcet criterion is violated.
2. Use the steps outlined in Ranked-Choice Voting for determining the winner of an election by ranked-choice voting, the application of the Hare method in which instant runoffs are used. **Step 1:** The number of votes needed to achieve a majority is 51.

Step 2: As illustrated in part 1, no candidate has a majority of first-place votes; so the candidate with the fewest votes, T, must be eliminated.

Step 3: Reallocate votes to the remaining candidates for the second round:

Votes	33	32	31	4
On a vacation (V)	1	3	3	2
Put into savings (S)	3	1	2	1
Pay off debt (D)	2	2	1	3

Step 4: Repeat the process from Step 2. Count the first-place votes for each candidate: V has 33 votes, S has 36 votes, D has 31. votes. Eliminate candidate D, "Pay off debt," for the third round.

Step 5: Repeat the process from Step 3. Reallocate votes to the remaining candidates for the third round:

Votes	33	32	31	4
On a vacation (V)	1	2	2	2
Put into savings (S)	2	1	1	1

Step 6: Repeat the process from Step 2 one last time. Count the first-place votes for each candidate: V has 33, S has 67. Candidate S, "Put into savings," has a majority and wins. Since the Condorcet candidate D candidate D, "Pay off debt," didn't win, the Condorcet criterion is violated.

1. Calculate the Borda score for each candidate. V: $33(4-1)+32(4-3)+31(4-3)+4(4-2)=170$

S: $33(4-3)+32(4-1)+31(4-2)+4(4-1)=203$

D: $33(4-2)+32(4-2)+31(4-1)+4(4-4)=223$

T: $33(4-4)+32(4-4)+31(4-4)+4(4-3)=4$

Candidate D, "Pay off debt," has the highest Borda score and wins. Since D was the Condorcet candidate, this election satisfies the Condorcet criterion.

As we have seen, the plurality method, ranked-choice voting, and the Borda count method each fail the Condorcet criterion in some circumstances. Of the four main ranked voting methods we have discussed, only the pairwise comparison method satisfies the Condorcet criterion every time. A summary of each voting method as it relates to the Condorcet criterion is found in the following table.

Voting Method	Condorcet Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Violates

Monotonicity Criterion

The citizens of Imaginaria might be surprised to learn that it is possible for a voter to cause a candidate to lose by ranking that candidate higher on their ballot. Is that fair? Most voters would say, "Absolutely not!!" This is an example of a violation of the fairness criterion called the **monotonicity criterion**, which is satisfied when no candidate is harmed by up-ranking nor helped by down-ranking, provided all other votes remain the same.

Consider a scenario in which voters are permitted a first round that is not binding, and then they may change their vote before the second round. Such a first round can be called a "straw poll." Now, let's suppose that a particular candidate won the straw poll. After that, several voters are convinced to increase their support, or **up-rank**, that winning candidate and no voters decrease that support. It is reasonable to expect that the winner of the first round will also win the second. Similarly, if some of the voters decide to decrease their support, or **down-rank**, a losing candidate, it is reasonable to expect that candidate will still lose in the second round.

You might be wondering why it's called the monotonicity criterion. In mathematics, the term monotonicity refers to the quality of always increasing or always decreasing. For example, a person's age is monotonic because it always increases, whereas a person's weight is not monotonic because it can increase or decrease. If the only changes to the votes for a particular candidate after a straw poll are in one direction, this change is considered monotonic.

If you are going to make an informed decision about which voting method to use in Imaginaria, you need to know which of the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—satisfy the monotonicity criterion.

Example 5 **Favorite Dog Breed by Plurality**

The local animal shelter is having a vote-by-donation charity event. For a $10 donation, an individual can complete a ranked ballot indicating their favorite large dog breed: standard poodle, golden retriever, Labrador retriever, or bulldog. Use the summary of ballots below to answer each question.

Votes	42	53	61	24
(S) Standard Poodle	1	3	2	1
(G) Golden Retriever	3	1	4	4
(L) Labrador Retriever	4	2	1	2
(B) Bulldog	2	4	3	3

1. Determine the winner of the election by plurality.
2. Suppose that the 53 voters in the second column increased their ranking of the winner by 1. Determine the winner by plurality with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is also true for plurality voting and the monotonicity criterion in general? Why or why not?

Solution

1. The number of votes for each candidate are: S 66, G 53, L 61, and B 0. The winner is the standard poodle.
2. If the 53 voters in the second column rank S as 2 and L as 3, then the number of votes for each candidate are: S with 66, G with 53, L with 61, and B with 0. The winner is still the standard poodle.
3. This election does not violate the monotonicity criterion because the winner was not hurt by up-ranking.

4. In general, increasing the ranking for a winner of a plurality election will either leave them with the same or more first place votes while leaving the other candidates with the same or fewer first place votes. So a plurality election will never violate the monotonicity criterion.

Example 6 **Monotonicity Criterion in a Pairwise Comparison Election**

Earlier, we discovered that the summary of ranked ballots shown in the table below results in the pairwise comparison matrix in . Use this information to answer the questions.

Number of Ballots	95	90	110	115
Option A	4	4	1	1
Option B	2	2	2	2
Option C	3	1	3	4
Option D	1	3	4	3

Opponent / Runner	Option A	Option B	Option C	Option D	Points
Option A wins	--	AB 225	AC 225	AD 225	3
Option B wins	~~BA 185~~	--	BC 320	BD 315	2
Option C wins	~~CA 185~~	~~CB 90~~	--	~~CD 200~~	0
Option D wins	~~DA 185~~	~~DB 95~~	DC 210	--	1

Figure 12: Analyzed Pairwise Comparison Matrix for Sample Summary of Ranked Ballots

1. Determine the winner of the election by the pairwise comparison method.
2. Suppose that the 95 voters in the first column increased their ranking of the winner by 1. Determine the winner by the pairwise comparison method with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part 3 is true for the pairwise comparison method and the monotonicity criterion in general? Why or why not?

Solution

1. By the pairwise comparison method, Option A wins with three points.
2. If the 95 voters in the first column of increased their ranking of the winner by 1, then C would fall into fourth place and A would move up to third place on those ballots. This would only affect the matchup between A and C, and the result would be that A

would gain 95 votes while C would lose 95 votes. This means A would have 320 votes and C would have 90. Since A already was ahead of C, this just puts A further ahead and causes no change to the election results.

3. Since the winner A is not hurt by an up-rank, and the loser C is not helped by a down-rank, this election is fair by the monotonicity criterion.
4. Yes, the monotonicity criterion would be satisfied by the pairwise comparison method, because an up-rank of the winner can never decrease the number of pairwise wins. Similarly, a down-rank can never increase the number of pairwise wins.

The last few examples illustrate that the plurality method, pairwise comparison voting, and the Borda count method each satisfy the monotonicity criterion. Of the four main ranked voting methods we have discussed, only the ranked-choice method violates the monotonicity criterion. A summary of each voting method as it relates to the Condorcet criterion is found in the table below.

Voting Method	Monotonicity Criterion
Plurality	Satisfies
Ranked-choice	Violates
Pairwise comparison	Satisfies
Borda count	Satisfies

Irrelevant Alternatives Criterion

We have covered a lot about voting fairness, but there is one more fairness criterion that you and the other Imaginarians should know. Consider this well-known anecdote that is sometimes attributed to the American philosopher Sidney Morgenbesser:

A man is told by his waiter that the dessert options this evening are blueberry pie or apple pie. The man orders the apple pie. The waiter returns and tells him that there is also a third option, cherry pie. The man says, "In that case, I would like the blueberry pie." (*Gaming the Vote: Why Elections Aren't Fair (and What We Can Do About It)*, William Pound stone, p. 50, ISBN 0-8090-4893-0)

This story illustrates the concept of the Irrelevant Alternatives Criterion, also known as the **Independence of Irrelevant Alternatives Criterion (IIA),** which means that the introduction or removal of a third candidate should not change or reverse the rankings of the original two candidates relative to one another. In particular, if a losing candidate is removed from the race or if a new candidate is added, the winner of the race should not change.

Example 7 **Apple, Blueberry, or Cherry?**

Suppose that 30 students in a class are going to vote on whether to have apple, blueberry, or cherry pie. Use the summary of ranked ballots in below to answer each question.

Number of Ballots	14	12	4
(A) Apple Pie	1	3	3
(B) Blueberry Pie	2	1	2
(C) Cherry Pie	3	2	1

1. Determine the winner of the election by plurality.
2. Which candidate would win a plurality election if cherry pie were removed from the ballot?
3. Does this election violate the IIA?

Solution

1. The number of first place votes for each candidate is: A with 14, B with 12, and C with 4. Apple pie has the most first-place votes and wins the election.
2. If cherry pie is removed from the ballot, then the four voters in the third column now rank blueberry pie as their first choice. So the four votes for C now belong to B. This means that blueberry pie has 16 votes compared to the 14 votes for apple pie. Blueberry pie now wins the plurality election.
3. Yes, the election violates the IIA because the removal of a losing candidate from the ballot changed the winner of the election.

Example 8 **Best Fourth Wall Breaking Stare on The Office**

The NBC sitcom *The Office* ran for nine years and has been one of the most popular streamed television shows of all time. One of the trademarks of the show was that characters would often break the fourth wall to communicate with the audience just by staring directly into the camera. In fact, there is a website dedicated to "*The Office* stares" where you can watch over 700 of these stares! Suppose that 36 fans were asked which character had the best "*The Office* stare." Use the ballot summary in below to answer each question.

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	4	2	4
(P) Pam Beesly-Halpert (Jenna Fischer)	4	1	2	4	3
(D) Dwight Schrute (Rainn Wilson)	2	3	3	1	2
(M) Michael Scott (Steve Carell)	3	4	1	3	1

1. Determine the winner of the election by the pairwise comparison method.

2. Determine the winner of the election by the pairwise comparison method if Michael Scott is removed from the ballot.
3. Does this election violate IIA?

Solution

1. Construct and analyze a pairwise comparison matrix:

Opponent Runner	J	P	D	M	Points
J wins	--	~~JP 15~~	JD 20	JM 26	2
P wins	PJ 21	--	~~PD 18~~	~~PM 11~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	~~DP 18~~	--	DM 26	$1\frac{1}{2}$
M wins	~~MJ 10~~	MP 25	~~MD 10~~	--	1

Figure 13: Pairwise Comparison Matrix for Jim, Pam, Dwight, and Michael

Jim Halpert wins with 2 points.

1. If Michael Scott is removed, the summary of ranked ballots becomes:

Number of Ballots	9	11	7	6	3
(J) Jim Halpert (John Krasinski)	1	2	3	2	3
(P) Pam Beesly-Halpert (Jenna Fischer)	3	1	1	3	2
(D) Dwight Schrute (Rainn Wilson)	2	3	2	1	1

Construct and analyze a pairwise comparison matrix:

Opponent Runner	J	P	D	Points
J wins	--	~~JP 15~~	JD 20	1
P wins	PJ 21	--	~~PD 18~~	$1\frac{1}{2}$
D wins	~~DJ 16~~	~~DP 18~~	--	0.5

Figure 14: Pairwise Comparison Matrix for Jim, Pam, and Dwight

Pam wins with $1\frac{1}{2}$ points.

1. Yes, this violates the IIA, because the winning candidate was hurt by the elimination of a losing candidate.

We have seen that all four of the main voting systems we are working with fail the Irrelevant Alternatives Criterion (IIA). A summary of each voting method as it relates to the IIA criterion is found in the table below.

Voting Method	Irrelevant Alternatives Criterion
Plurality	Violates
Ranked-choice	Violates
Pairwise comparison	Violates
Borda count	Violates

WHO KNEW?

Electronic Voting: Does Your Vote Count?

In order for an election to be fair, voting must be accessible to everyone and every vote must be counted. When hundreds of thousands to millions of votes must be collected and counted in a short period of time, deciding it can be challenging to be on counting procedures that are accurate and secure. Electronic voting machines or even Internet voting can speed up the process, but how reliable are these methods? This has been a subject for debate for years.

In a press release on August 3, 2007, California Secretary of State Debra Bowen explained the results of an extensive review of electronic voting systems in her state. She said that transparency and auditability were key. She went on to say, “I think voters and counties are the victims of a federal certification process that hasn’t done an adequate job of ensuring that the systems made available to them are secure, accurate, reliable and accessible. Congress enacted the Help America Vote Act, which pushed many counties into buying electronic systems that—as we’ve seen for some time and we saw again in the independent UC review—were not properly reviewed or tested to ensure that they protected the integrity of the vote.” Secretary Bowden subsequently ordered that voting machines must have tighter security to be used in California. (DB07:042, Secretary of State Debra Bowen Moves to Strengthen Voter Confidence in Election Security Following Top-to-Bottom Review of Voting Systems, https://sos.ca.gov/elections.) In some instances, the use of electronic voting in parliamentary elections has been discontinued completely for security reasons. For example, according to the National Democratic Institute, the Netherlands returned to all paper ballots and hand counting in 2006. Will you use voting machines or Internet voting in Imaginaria?

So far, every one of the voting methods we have analyzed has failed one or more of the fairness criteria in one election or another.

Voting Method	Majority Criterion	Condorcet Criterion (Head-to-Head Criterion)	Monotonicity Criterion	(Independence of) Irrelevant Alternatives Criterion
Plurality	Satisfies	Violates	Satisfies	Violates
Ranked-choice	Satisfies	Violates	Satisfies	Violates
Pairwise comparison	Satisfies	Satisfies	Violates	Violates
Borda count	Violates	Violates	Satisfies	Violates

You might be wondering if there is a voting system you could recommend for Imaginaria that satisfies all the fairness criteria. If there is one, it remains to be discovered and it is not a voting system that is based solely on preference rankings. In 1972, Harvard Professor of Economics Kenneth J. Arrow received the Nobel Prize in Economics for proving **Arrow's Impossibility Theorem**, which states that any voting system, either existing or yet to be created, in which the only information available is the preference rankings of the candidates, will fail to satisfy at least one of the following fairness criteria: the majority criterion, the Condorcet criterion, the monotonicity criterion, and the independence of irrelevant alternatives criterion. This theorem only applies to a specific category of voting systems—those for which the preference ranking is the only information collected. There are other types of voting systems to which the Impossibility Theorem does not apply. For example, there is a class of voting systems called **Cardinal voting systems** that allow for rating the candidates in some way.

"Rating" is different from "ranking" because a voter can give different candidates the same rating. Consider the five-star rating systems used by various industries, or the thumbs up/thumbs down rating system used on YouTube. Could there be a Cardinal voting system that does not violate any of the fairness criteria we have discussed? It's possible, but more research must be done in order to prove it.

PEOPLE IN MATHEMATICS

Kenneth J. Arrow

In 1972, Kenneth J. Arrow, a Harvard Professor of Economics, received the Nobel Prize in Economics jointly with Sir John Hicks, another world renowned economist, for their contributions to economic theory. In particular, Professor Arrow proved mathematically that no ranked voting system meets all four of the fairness criteria discussed in this section. The statement of this fact is known as Arrow's Impossibility Theorem. Visit this site for more details on Professor Arrow

Key Terms

- majority criterion
- tyranny of the majority
- Condorcet criterion
- Condorcet method
- monotonicity criterion
- up-rank
- down-rank
- independence of irrelevant alternatives criterion (IIA)
- Arrow's Impossibility Theorem
- cardinal voting system

Key Concepts

- There are several common measures of voting fairness, including the majority criterion, the head-to head criterion, the monotonicity criterion, and the irrelevant alternatives criterion.
- According to Arrow's Impossibility Theorem, each voting method in which the only information is the order of preference of the voters will violate one of the fairness criteria.

Videos

- Separation of Powers and Checks and Balances

11.3 Standard Divisors, Standard Quotas, and the Apportionment Problem

Figure 15: Every person at a party gets their fair slice of the cake.

Learning Objectives

After completing this section, you should be able to:

1. Analyze the apportionment problem and applications to representation.
2. Evaluate applications of standard divisors.
3. Evaluate applications of standard quotas.

The Apportionment Problem

In the new democracy of Imaginaria, there are four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Each state will have representatives in the Imaginarian Legislature. You might now have an agreement on which voting method your citizens will use to elect representatives. However, before that process can even begin, you must decided on how many representatives each state will receive. This decision will present its own challenges.

When sharing your birthday cake, it's only fair that everyone gets the same portion size, right? You were portioning the cake by dividing it up equally and giving everyone a slice. A great thing about cake is that you can slice it any way you want, but how do you **apportion**, or divide and distribute, items that can't be sliced? Suppose that you have a box of 16 Ring Pops™, gem-shaped lollipops on a plastic ring. You are going to share the box with four other kids. Dividing the 16 Ring Pops™ among the group of five leads to a problem; after each person in the group gets three Ring Pops™, there is still one left! Who gets the last one? The **apportionment problem** is how to fairly divide, or apportion, available resources that must be distributed to the recipients in whole, not fractional, parts.

The apportionment problem applies to many aspects of life, including the representatives in the Imaginarian legislature. The table below provides a short list of examples of resources that must be apportioned in whole parts, and the recipients of those resources.

Resource	Recipients
Covid-19 Vaccines	Nations around the world
Airport Terminals	Airlines
Faculty Positions at a University	Departments
Public Schools	Communities
U.S. House of Representatives Seats	States
Parliamentary Seats	Political Parties

Fair division of a resource is not necessarily equal division of the resource like when distributing cake slices. When distributing airport terminals amongst airlines, there are many factors to consider such as the size of the airline, the number and types of aircraft they have, and the demand for the service. In most cases, fairness is defined as being **proportional** two quantities are proportional if they have the same relative size. In the case of the Covid-19 vaccine, the expectation would be that countries with larger populations get more doses of the vaccine. In the Imaginarian legislature, the expectation may be that the states with larger populations will receive the larger number of representatives. This concept is referred to as a **part-to-part ratio**.

Suppose that a supermarket has a special on pies, two for \$5. The first customer purchases four pies for \$10, and the second customer purchases eight pies for \$20. The dollar to pie ratio for the first customer is $\frac{10 \text{ dollars}}{4 \text{ pies}} = 2.5$ dollars per pie and the dollar to pie ratio for the second customer is $\frac{20 \text{ dollars}}{5 \text{ pies}} = 2.5$ dollars per pie. So, the dollar to pie ratio is constant. Although the customers do not spend the same amount of money, the amount each spent was proportional to the number of pies purchased.

Now suppose that the supermarket changed the special to \$5 for the first pie, and \$2 for each additional pie. In that case, four pies would cost $\$\,5 + 3(\$\,2) = \$\,11$, while 8 pies would cost $\$\,5 + 7(\$\,2) = \$\,19$. The dollar to pie ratios would be $\frac{11 \text{ dollars}}{4 \text{ pies}} = 2.75$ dollars per pie and $\frac{19 \text{ dollars}}{8 \text{ pies}} = 2.375$ dollars per pie, respectively. This special does not result in a constant part to part ratio. The dollars spent are not proportional to the number of pies purchased.

VIDEO

What Is a Ratio?

What Are the Different Types of Ratios?

Example 1 **Ratio of Faculty to Students at a College**

The following table provides a comparison of the number of faculty members in each department at a particular college to the student head count in that department and the number of class sections in that department in the Spring semester. Use this information to answer the questions.

Department	Mathematics	English	History	Science
(S) Student Head Count	4800	2376	1536	2880
(C) Class Sections	120	108	48	96
(T) Total Faculty	30	27	12	24
(F) Full-Time Faculty	10	9	4	8
(P) Part-Time Faculty	20	18	8	16

1. Determine the ratios for each department: S to C, C to T, S to T, F to P
2. What are the units of the ratios that you found?
3. Which of these pairs, if any, has a constant part to part ratio? State the ratio.
4. Does it appear that the total number of faculty positions were allocated to each department based on student head count, the number of class sections, or neither? Justify your answer.

Solution

1. Divide the first quantity by the second, for each department, as shown in the table below.

2. Answers are provided in last column of the table.

Department	Mathematics	English	History	Science	Units of Ratios Found
S to C	$\frac{4800}{120} = 40$	$\frac{2376}{108} = 22$	$\frac{1536}{48} = 32$	$\frac{2880}{96} = 30$	Students per class section
C to T	$\frac{120}{30} = 4$	$\frac{108}{27} = 4$	$\frac{48}{12} = 4$	$\frac{96}{24} = 4$	Class sections per faculty member
S to T	$\frac{4800}{30} = 160$	$\frac{2376}{27} = 88$	$\frac{1536}{12} = 128$	$\frac{2880}{24} = 120$	Students per faculty member
F to P	$\frac{10}{20} = \frac{1}{2}$	$\frac{9}{18} = \frac{1}{2}$	$\frac{4}{8} = \frac{1}{2}$	$\frac{8}{16} = \frac{1}{2}$	Full-time faculty member per part-time faculty member

3. The ratio of class sections to faculty members is a constant ratio of four. The ratio of full-time faculty to part-time faculty is a constant ratio of $\frac{1}{2}$.
4. It appears that the faculty positions were allocated based on the number of class sections because there is a constant ratio of four class sections per faculty member.

There are some useful relationships between quantities that are proportional to each other. When there is a constant ratio between two quantities, the one quantity can be found by multiplying the other by that ratio. Remember the supermarket special on pies, 2 pies for \$5? The ratio of dollars to pies is $\frac{5\text{ dollars}}{2\text{ pies}} = 2.5$ dollars per pie and the ratio of pies to dollars is $\frac{2\text{ pies}}{5\text{ dollars}} = 0.4$ pies per dollar. These two values are reciprocals of each other, $\frac{1}{2.5} = 0.4$ and $\frac{1}{0.4} = 2.5$. This means that multiplying by one has the same effect as dividing by the other. This also means that knowing either constant ratio allows us to calculate the price given the number of pies. To find the cost of 20 pies, multiply by the ratio of dollars to pies or divide by the ratio of pies to dollars.

- 20 pies $\times$ 2.50 dollars per pie $=$ 50 dollars
- 20 pies $\div$ 0.4 pies per dollar $=$ 50 dollars

  These patterns are true in general.

FORMULA

Let A be a particular item and B another such that there is a constant ratio of A to B

- ratio of B's to A's $= \frac{1}{\text{ratio of } A\text{'s to } B\text{'s}}$ and ratio of A's to B's $= \frac{1}{\text{ratio of } B\text{'s to } A\text{'s}}$
- units of A = (units of B) $\times$ (ratio of A's to B's) $= \frac{\text{units of } B}{\text{ratio of } B\text{'s to } A\text{'s}}$
- units of B = (units of A) $\times$ (ratio of B's to A's) $= \frac{\text{units of } A}{\text{ratio of } A\text{'s to } B\text{'s}}$

Example 2 **Ratio of Faculty to Students at a College**

Refer to the information given in Example 1.

1. If there are 32 class sections each semester in the Fine Art department, and the same ratio is used to determine the number of faculty members, how many faculty members would you expect to see in the Fine Art department?
2. If the Health Sciences department has 6 full-time faculty members, how many part-time faculty members are in the department?

Solution

1. Multiply the number of class sections by the ratio of faculty members to class sections to find the number of faculty. Since there are 4 classess per faculty member, the ratio of faculty members to classs sections is $\frac{1}{4}$. This means that the number of faculty members for 32 classes should be $32 \times \frac{1}{4} = 8$ faculty members.
2. Multiply the number of full-time faculty by the ratio of part-time to full-time to find the number of part-time. Since ratio of full-time faculty to part-time faculty at the college is $\frac{1}{2}$ or 1 full-time per 2 part time, the ratio of part-time to full-time is $\frac{2}{1} = 2$ part-time to 1 full-time; so the number of part-time faculty in a department with 6 full-time faculty should be $6.4 = 24$ part-time faculty.

The apportionment application that will be important to the founders of Imaginaria occurs in **representative democracies** in which elected persons represent a group. The United Kingdom, France, and India each have a parliament, and the United States has a Congress, just as Imaginaria will have a legislature! The citizens of a country must decide what portion of the representatives each group, such as a state or province or even a political party, will have. A larger portion of representatives means greater influence over policy.

Example 3 **Ratio of U.S. Representatives to State Population**

contains a list of the five U.S. states with the greatest number of representatives in the U.S. House of Representatives, along with the population of that state in 2021. Use the information in the table to answer the questions.

State	Representative Seats	State Population
(CA) California	53	39,613,000
(TX) Texas	36	29,730,300
(NY) New York	27	19,300,000
(FL) Florida	27	21,944,600
(PA) Pennsylvania	18	12,804,100

1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
4. Does there appear to be a constant ratio? Justify your answer.

Solution

1. CA 700,000; TX 800,000; NY 700,000; FL 800,000; PA 700,000
2. CA 0.0000013; TX 0.0000012; NY 0.0000014; FL 0.0000012; PA 0.0000014
3. CA 0.000001; TX 0.000001; NY 0.000001; FL 0.000001; PA 0.000001
4. The ratio of State Population to Representative Seats seems to be either 700,000 or 800,000. There does appear to be a constant ratio of about 0.000001 of Representative Seats to State Population if we round off to the sixth decimal place.

VIDEO

Math Antics – Rounding

You might be wondering why the ratio doesn't appear to be quite the same depending on the rounding of the values. We will see that the key to this variation lies in the fractions. Just like the five children sharing 16 Ring Pops™, there are going to be leftovers and there are many methods for deciding what to do with those leftovers.

The Standard Divisor

There are two houses of congress in the United States: the Senate and the House of Representatives. Each state has two senators, but the number of representatives depends on the population of the state. The number of representative seats in the U.S. House of Representatives is currently set by law to be 435. In order to distribute the seats fairly to each state, the ratio of the population of the U.S. to the number of representative seats must be calculated. The ratio of the total population to the house size is called the **standard divisor**, and it is the number of members of the total population represented by one seat.

Although apportionment applies to many other scenarios, such as the pencil distribution during the SAT, the terminology of apportionment is based on the House of Representatives scenario. Thus, several government-related terms take on a more general meaning. The **states** are the recipients of the apportioned resource, the **seats** are the units of the resource being apportioned, the **house size** is the total number of seats to be apportioned, the **state population** is the measurement of the state's size, and the **total population** is the sum of the state populations.

FORMULA

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{House Size}}$$

Example 4 **The Standard Divisor of the U.S. House of Representatives 2021**

As of this writing, the Census.gov website U.S. Population clock showed a population of 332,693,997. There are 435 seats in the U.S. House of Representatives. Find the standard divisor rounded to the nearest tenth.

Solution

Dividing $330,147,881$ people by 435 seats, there are 758,960.6 people per representative.

Whether the standard divisor is less than, equal to, or greater than 1 depends on the ratio of the population to the number of seats.

- The standard divisor will be equal to 1 if the total population is equal to the number of seats. This would mean that each member of the population is allocated their own personal seat.
- The standard divisor will be a number between 0 and 1 when the total population is less than the number of seats. This means that each member of the population is allocated more than one seat.
- The standard divisor will be a number greater than 1when the total population is greater than the number of seats. This means that a certain number of members of the population will share 1 seat.

  If the population is five children and the house consists of five pieces of candy, the standard divisor is $\frac{5 \text{ children}}{5 \text{ candies}} = 1$ child per candy meaning each child gets one candy. If the population is five children and ten pieces of candy, the standard divisor is $\frac{5 \text{ children}}{10 \text{ candies}} = 0.5$ child per candy meaning that each child gets more than one candy. If the population is five children and four pieces of candy, the standard divisor is $\frac{5 \text{ children}}{4 \text{ candies}} = 1.25$ child per candy meaning that each child gets less than one candy.

  If the seats in the Imaginarian legislature are distributed to the states based on population, then the house size will be less than the population and we should expect the standard divisor to be a number greater than 1.

Example 5 **School Resource Officers in Brevard County, Florida**

The public schools in a certain county have been allotted 349 school resource officers to be distributed among 327 public schools attended by approximately 271,500 students.

1. Identify the states, seats, house size, state population, and total population in this apportionment scenario.
2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to the nearest tenth.

Solution

1. The states are the schools in that county. The seats are the school resource officers. The house size is the number of school resource officers, which is 349. The state population is the number of students in a particular school, which was not given. The total population consists of the sum of the school populations, which is 271,500.
2. The standard divisor is the ratio of the total population to the house size, which is the number of students served by each resource officer. Divide $271,500$ students $\div$ 349 officers $= 777.9$ students per officer.

The Standard Quota

Once the standard divisor for the Imaginarian legislature is calculated, the next task is to determine the number of seats that each state should receive, which is referred to as the state's **standard quota**. Unless all the states have the same population, each state will receive a different number of seats because the quantities will be proportionate to the state populations. To determine those amounts, we will use an idea we learned earlier. Recall that, when the number of units of item A is proportionate to the number of units of item B, we have: units of $A = \frac{\text{units of } B}{\text{ratio of } B\text{'s to } A\text{'s}}$

In this case, we are trying to calculate the number of seats a state should be apportioned, the state's standard quota. So A So A would refer to seats allocated to a particular state, while B would refer to the state population. This means that the ratio of B to A is the ratio of the total population to house size, which is the standard divisor. So in apportionment terms, we have the following formula.

FORMULA

$$\text{State's Standard Quota} = \frac{\text{State Population}}{\text{Standard Divisor}} \text{ seats}$$

Example 6 **The Standard Quota of the U.S. House of Representatives 2021**

Example 27 outlined that the Census.gov website U.S. Population clock showed a population of 330,147,881, there are 435 seats in the U.S. House of Representatives, and the standard divisor was 758,960.6 people per representative. The state of California has a population of approximately 39,613,000. Use these values to determine the standard quota for California to two decimal places.

Solution

$$\text{California's standard Quota} = \frac{\text{California's population}}{\text{Standard Divisor}} = \frac{39,613,000}{758,960.6} = 52.19 \text{ representatives}$$

Example 7 **Apportionment of Laptops in a Science Department**

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom's student capacity. Use the values in the table below to find the standard quota for each classroom.

Room	Students
A	30
B	25
C	28
D	32
E	24
F	27

Solution

Step 1: Identify the state population, total population, and the house size. The states are the classrooms, and the state populations are listed in the table. The total population is the sum of the state populations, which is 166. The house size is the number of seats, or laptops, to be allocated, which is 34.

Step 2: Calculate the standard divisor by dividing the total population by the house size.

$$\text{Standard Divisor} = \frac{\text{Total of Room Capacities}}{\text{Number of Laptops}} = \frac{166}{34} \approx 4.88$$

Step 3: Calculate the standard quota by dividing the state population by the standard divisors , as shown in the table below. Room's Standard Quota $= \frac{\text{Room Capacity}}{\text{Standard Divisor}}$

Room	Room Capacity	Room's Standard Quota
A	30	$30 \div 4.88 \approx 6.15$ laptops
B	25	$25 \div 4.88 \approx 5.12$ laptops
C	28	$28 \div 4.88 \approx 5.74$ laptops
D	32	$32 \div 4.88 \approx 6.56$ laptops
E	24	$24 \div 4.88 \approx 4.92$ laptops
F	27	$27 \div 4.88 \approx 5.53$ laptops

Step 4: Find the sum of the standard quotas. $6.15 + 5.12 + 5.74 + 6.56 + 4.91 + 5.53 = 34.01$. This is only slightly off from the number of laptops—34—which can be caused by rounding off in previous steps. This is a good indication that the calculations were correct. If you find that the value of the sum of the standard quotas is significantly different from the house size (number of seats), it is possible that the standard divisor was calculated using too few decimal places. Calculate the standard divisor and standard quotas again but round off to a greater number of decimal places.

Key Terms

- apportion
- apportionment problem
- proportional
- part-to-part ratio
- representative democracies
- standard divisor
- states
- seats
- house size
- state population
- total population
- standard quota

Key Concepts

- The apportionment problem is how to fairly divide and distribute available resources to recipients in whole, not fractional, parts.

- To distribute the seats in the U.S. House of Representatives fairly to each state, calculations are based on state population, total population, and house size, or the total number of seats to be apportioned.
- The standard divisor is the ratio of the total population to the house size, and the standard quota is the number of seats that each state should receive.

Formulas

Let A be a particular item and B another such that there is a constant ratio of A to B.

- ratio of B's to A's $= \frac{1}{\text{ratio of } A\text{'s to } B\text{'s}}$ and ratio of A's to B's $= \frac{1}{\text{ratio of } B\text{'s to } A\text{'s}}$
- units of A = (units of B) $\times$ (ratio of A's to B's) $= \frac{\text{units of } B}{\text{ratio of } B\text{'s to } A\text{'s}}$
- units of B = (units of A) $\times$ (ratio of B's to A's) $= \frac{\text{units of } A}{\text{ratio of } A\text{'s to } B\text{'s}}$

$$\text{Standard Divisor} = \frac{\text{Total Population}}{\text{House Size}}$$

$$\text{State's Standard Quota} = \frac{\text{State Population}}{\text{Standard Divisor}} \text{ seats}$$

Videos

- What Is a Ratio?
- What Are the Different Types of Ratios?
- Math Antics – Rounding

11.4 Apportionment Methods

Figure 16: Schoolchildren depend on apportionment of resources like laptops among schools and classroom.

Learning Objectives

After completing this section, you should be able to:

1. Describe and interpret the apportionment problem.
2. Apply Hamilton's Method.
3. Describe and interpret the quota rule.
4. Apply Jefferson's Method.
5. Apply Adams's Method.
6. Apply Webster's Method.
7. Compare and contrast apportionment methods.
8. Identify and contrast flaws in various apportionment methods.

A Closer Look at the Apportionment Problem

In Standard Divisors, Standard Quotas, and the Apportionment Problem we calculated the standard divisor and the standard quotas in various apportionment scenarios. The results of those calculations routinely led to fractions and decimals of units. However, the seats in the House of Representatives, laptops in a classroom, or a variety of other resources, are indivisible, meaning they cannot be divided up into fractional parts. This leaves a decision to be made. For example, if the standard quota for the number of laptops to be distributed to a classroom is 12.44 units, how do we deal with the fractional part of 0.44? It is unclear if the classroom should receive 12 units, 13 units, or some other value. Let's try traditional rounding to the nearest whole number value.

Example 1 **Installing Emergency Lights**

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in three parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which is to be measured in acres. The total number of acres is 34; so the standard divisor is $\frac{34}{70} \approx 0.4857$. The standard quota for each lot is listed in the table below. Use this information to answer each question.

Lot	Acres	Lot's Standard Quota
A	15	$15 \div 0.4857 \approx 30.88$ emergency blue lights
B	9	$9 \div 0.4857 \approx 18.53$ emergency blue lights
C	10	$10 \div 0.4857 = 20.59$ emergency blue lights

1. Use traditional rounding to determine the number of lights assigned to each lot.
2. Find the sum of the values from part 1.
3. Does the sum found in part 2 equal the number of lights available?

Solution

1. If traditional rounding is used, there will be 31, 19, and 21 lights distributed to each lot, respectively.

2. The total of these values is 71.
3. No, the total from part 2 is one more than the number of lights available. In other words, one of the parking lots must get 1 fewer light than apportioned.

Example 1 demonstrates that we cannot successfully apportion indivisible resources by rounding off each standard quota using traditional rounding. This leaves us with a problem. What is a fair way to distribute the fractional parts of the standard quotas? We will refer to this as the **apportionment problem**. Several methods for making this decision will be discussed.

Example 2 **A-10C Thunderbolt II Aircraft**

In 2015, the U.S. Air Force had a fleet of approximately 281 A-10C Thunderbolt II aircraft. Suppose that the Air Force administration wanted to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. Use the information in the table below to answer each question.

Base	Pilots
(A) Alpha	13
(B) Bravo	12
(C) Charlie	5
(D) Delta	16
(E) Echo	7
(F) Foxtrot	9

1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.
2. Find the standard divisor for the apportionment of the aircraft. Round to four decimal places as needed. Include the units.
3. Find each Air Force base's standard quota for the apportionment of the aircraft. Round to the nearest hundredth as needed. What are the units?
4. How does this example demonstrate the apportionment problem? Will traditional rounding solve the problem?

Solution

1. The states are the bases, the seats are the aircraft, and the state populations are the pilots at a given base.
2. Standard Divisor $= \frac{\text{Total Population}}{\text{House Size}} = \frac{13+12+5+16+7+9}{27} = \frac{62}{27} \approx 2.2963$ pilots per aircraft.

3. A $\frac{13}{2.2963} \approx 5.66$, B $\frac{12}{2.2963} \approx 5.23$, C $\frac{5}{2.2963} \approx 2.18$, D $\frac{16}{2.2963} \approx 6.97$, E $\frac{7}{2.2963} \approx 3.05$, F $\frac{9}{2.2963} \approx 3.92$. The units are aircraft.
4. This example demonstrates the apportionment problem because it is not possible to send a fractional number of aircraft to an Air Force base. On the other hand, if we use traditional rounding methods to get whole numbers, the results are $5 + 5 + 2 + 7 + 3 + 4 = 26$ aircraft will be apportioned, which is one less than the number of aircraft that were supposed to be apportioned.

Hamilton's Method of Apportionment

One of the problems encountered when standard quotas are transformed into whole numbers using traditional rounding is that it is possible for the sum of the values to be greater than the number of seats available. A reasonable way to avoid this is to always round down, even when the first decimal place is five or greater. For example, a standard quota of 12.33 and a standard quota of 12.99 would both round down to 12. This is called the **lower quota**.

Example 3 **Lower Quota for Apportionment of Aircraft**

The Air Force administration wants to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. The standard quotas for each base are listed in the table below. Use this information to answer the questions.

Base	Standard Quota
(A) Alpha	$\frac{13}{2.2963} \approx 5.66$ aircraft
(B) Bravo	$\frac{12}{2.2963} \approx 5.23$ aircraft
(C) Charlie	$\frac{5}{2.2963} \approx 2.18$ aircraft
(D) Delta	$\frac{16}{2.2963} \approx 6.97$ aircraft
(E) Echo	$\frac{7}{2.2963} \approx 3.05$ aircraft
(F) Foxtrot	$\frac{9}{2.2963} \approx 3.92$ aircraft

1. Give the lower quota for each Air Force base.
2. Find the sum of the lower quotas. By how much does this sum fall short of the actual number of aircraft?

Solution

1. Round down. The lower quota for each Air Force base is 5, 5, 2, 6, 3, 3, respectively.
2. The sum is 24. This is 3 fewer than the actual number of aircraft.

If the standard quotas are all rounded down, their sum will always be less than or equal to the house size. Then, it would only remain to find a fair way to distribute any remaining seats.

Alexander Hamilton, who was a general in the American Revolution, author of the Federalist Papers, and the first secretary of the treasury, took this approach to apportionment.

Steps for Hamilton's Method of Apportionment

There are five steps we follow when applying Hamilton's Method of apportionment:

1. Find the standard divisor.
2. Find each state's standard quota.
3. Give each state the state's lower quota (with each state receiving at least 1 seat).
4. Give each remaining seat one at a time to the states with the largest fractional parts of their standard quotas until no seats remain.
5. Check the solution by confirming that the sum of the modified quotas equals the house size.

VIDEO

Hamilton Method of Apportionment

Example 4 **Hawaiian School Districts**

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under 19 years old, This data is provided in the table below. Using the Hamilton method, calculate how many schools would be funded in each state.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

Step 1: Calculate the standard divisor. Divide the total population, 325,570, by the house size, 616 seats. The standard divisor is 528.52.

Step 2: Find each state's standard quota:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx$ 87.62	$\frac{224,230}{528.52} \approx$ 424.26	$\frac{20}{528.52} \approx$ 0.04	$\frac{16,560}{528.52} \approx$ 31.33	$\frac{38,450}{528.52} \approx$ 72.75	616

Step 3: Find each state's lower quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Lower Quota	87	424	1	31	72	615

Step 4: Compare the sum of the states' lower quotas, 615, to the house size, 616. One seat remains to be apportioned and must be given to the state with the largest fractional part: Maui with 0.75. So, the final Hamilton quotas are as follows: Hawaii 87, Honolulu 424, Kalawao 1, Kauai 31, and Maui 73.

Step 5: Find the total to confirm the sum of the quotas equals the house size, 616. Then $87 + 424 + 1 + 31 + 73 = 616$. The apportionment is complete.

TECH CHECK

Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Hamilton apportionment calculator.

This can be a useful tool to confirm your results!

The Quota Rule

A characteristic of an apportionment that is considered favorable is when the final quota values all either result from rounding down or rounding up from the standard quotas. The value that results from rounding down is called the lower quota, and the value that results from rounding up is called the **upper quota**.

As we explore more methods of apportionment, we will consider whether they satisfy the quota rule. If a scenario exists in which a particular apportionment allocates a value greater than the upper quota or less than the lower quota, then that apportionment violates the quota rule and the apportionment method that was used violates the quota rule.

Example 5 **Which Apportionment Method Satisfies the Quota Rule?**

Several apportionment methods have been used to allocate 125 seats to ten states and the results are shown in the table below. Determine which apportionments do not satisfy the quota rule and justify your answer.

	State A	State B	State C	State D	State E	State F	State G
Standard Quota	41.26	16.00	5.77	2.64	7.82	10.47	0.21
Lower Quota	41	16	5	2	7	10	0
Upper Quota	42	17	6	3	8	11	1
Method X	43	16	5	2	7	10	1
Method Y	41	16	6	2	8	10	1
Method Z	42	16	7	3	7	9	1

Solution

Look for states such that the number of seats allocated differs from the lower or upper quota. Method X violates the quota rule because State A receives 43 seats instead of 41 or 42. Method Z violates the quota rule because State C receives 7 seats instead of 5 or 6 and State F receives 9 instead of 10 or 11.

It is possible for an apportionment method to satisfy the quota rule in some scenarios but violate it in others. However, because the Hamilton method always begins with the lower quota and either adds one to it or keeps it the same, the final Hamilton quota will always consist of values that are either lower quota values or upper quota values. When an apportionment method has this characteristic, it is said to satisfy the quota rule. So, we can say:

The Hamilton method of apportionment satisfies the quota rule.

Although the Hamilton method of apportionment satisfies the quota rule, it can result in some unexpected outcomes, which has caused it to pass in and out of favor of the U.S. government over the years. There are several apportionment methods that have been popular alternatives, such as Jefferson's method of apportionment that the founders of Imaginaria should consider.

Jefferson's Method of Apportionment

Another approach to dealing with the fractional parts of the standard quotas is to modify the standard divisor so that the total of the resulting modified lower quotas is the necessary number of seats. This is the approach used by Jefferson.

In Jefferson's method, the change to the standard divisor is made so that the total of the modified lower quotas equals the house size. The change in the standard divisor to get the modified divisor is relatively small. There is not a formula for this. The modified divisor is found by "guess and check." It is important to remember that *increasing* the divisor *decreases* the quotas, but *decreasing* the divisor *increases* the quotas. So, if you need a larger quota, try reducing the divisor, and if you need a smaller quota, try increasing the divisor.

Example 6 **Modifying a Standard Divisor**

Suppose the population of a state is 50 and the standard divisor is 12.5.

1. Find the state's standard quota.
2. Increase the standard divisor by 2 units and use the modified divisor to determine the modified quota for the state.
3. Decrease the modified divisor from part 2 by 1.5 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from part 2 and the value of the modified divisor from part 3 and use it to determine the modified quota for the state.
5. Which modified quota was the largest, the modified quota from part 2, from part 3, or from part 4? Explain why.

Solution

1. The state's standard quota is $\frac{50}{12.5} = 4$.
2. The modified divisor is 14.5. The modified quota is $\frac{50}{14.5} \approx 3.45$.
3. The modified divisor is 13. The modified quota is $\frac{50}{13} \approx 3.85$.
4. One value between 13 and 14.5 is 13.5. With a modified divisor of 13.5, the modified quota is $\frac{50}{13.5} \approx 3.70$.
5. The modified quota from part 3 was the largest because the divisor was the smallest of the three. Dividing the same number by a smaller value gives a larger result.

When you use Jefferson's method, you might have to adjust the divisor several times find modified lower quotas that sum to the house size. First, guess what the divisor should be based on the sum of the lower quotas and then increase or decrease it from there based on whether the sum needs to be smaller or larger respectively. If the result still does not produce lower quotas that sum to the house size, adjust again. Keep a record of the values that didn't work to help you narrow your search.

Steps for Jefferson's Method of Apportionment

We take four steps to apply Jefferson's Method of apportionment:

Step 1: Find the standard divisor.

Step 2: Find each state's quota. This will be the standard quota the first time Step 2 is completed and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.

Step 3: Find the states' lower quotas (with each state receiving at least one seat), and their sum.

Step 4: If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the lower quotas is less than the number of seats, reduce the standard divisor. If the sum of the lower quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

Example 7 **Hawaiian State Representative Districts**

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of five counties based on the residents under the age of 19. With the data in the table below, apply Jefferson's method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

Step 1: The process for finding the standard divisor, standard quotas, and lower quotas is the same in the Hamilton and Jefferson methods of apportionment. We walked through the Hamilton Method in Example 4, and following these steps resulted in lower quotas as shown in the table below.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Lower Quota	87	424	1	31	72	615

Step 2: Compare the sum of the states' lower quotas, 615, to the house size, 616. Since 615 is less than 616, use a modified divisor that is less than the standard divisor of 528.52. Try 526.00.

Step 3: Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{526.00} \approx 88.04$	$\frac{224,230}{526.00} \approx 426.29$	$\frac{20}{526.00} \approx 0.04$	$\frac{16,560}{526.00} \approx 31.48$	$\frac{38,450}{526.00} \approx 72.75$	616
Lower Quota	88	426	1	31	72	618

Step 4: The new sum of the lower quotas is 2 units greater than 616. We have overshot the goal. So, increase the divisor to a value between 526.00 and 528.52. Try 527.00.

Step 5: Repeat the process of finding the quotas. Find each state's modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{527.00} \approx 87.87$	$\frac{224,230}{527.00} \approx 425.48$	$\frac{20}{527.00} \approx 0.04$	$\frac{16,560}{527.00} \approx 31.42$	$\frac{38,450}{527.00} \approx 72.96$	616
Lower Quota	87	425	1	31	72	616

Step 6: The new sum of the lower quotas equals the house size. The apportionment is complete.

The apportionment is: Hawaii County 87, Honolulu County 425, Kalawao County 1, Kauai 31, and Maui 72 schools.

When using Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Notice that, in this apportionment, Mythbury received more than the upper quota. Since this apportionment of representatives to Imaginarian states by Jefferson's method does not satisfy the quota rule, we say that:

Jefferson's method violates the quota rule.

We have discussed two apportionment methods: one that satisfies the quota rule and one that does not. Before you decide which method to use in Imaginaria, there are a couple more options to consider.

VIDEO

Jefferson Apportionment Method

TECH CHECK

Apportionment Calculators

It is possible to create Excel spreadsheets that complete the calculations necessary to complete a Jefferson Apportionment. In some cases, this work has already been done and posted online. Check out websites such Ms. Hearn Math for a free Jefferson apportionment calculator.

This can be a useful tool to confirm your results!

Adams's Method of Apportionment

Adams's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas.

To apply Adams's Method of apportionment, there are four steps we follow:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Find the states' upper quotas and their sum.
4. If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the upper quotas is less than the number of seats, reduce the standard divisor. If the sum of the upper quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

Example 8 **Hawaiian School Districts**

As in earlier examples, suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under the age of

19. Use the data in the following table and the Adams method to apportion the schools to the counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

Step 1: The steps of finding the standard divisor and each state's quota are the same in the Jefferson and Adams methods. As in Example 7, the standard divisor is 528.52.

Step 2: Find each state's upper quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Upper Quota	88	425	1	32	73	619

Step 3: Compare the sum of the states' upper quotas, 619, to the house size, 616. Since 619 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{534.00} \approx 86.72$	$\frac{224,230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16,560}{534.00} \approx 31.01$	$\frac{38,450}{534.00} \approx 72.00$	616
Upper Quota	88	420	1	32	72	613

Step 5: The new sum of the upper quotas is 3 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 532.00.

Step 6: Find each state's modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 532.00:

County	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{532.00} \approx 87.05$	$\frac{224,230}{532.00} \approx 421.48$	$\frac{20}{532.00} \approx 0.04$	$\frac{16,560}{532.00} \approx 31.13$	$\frac{38,450}{532.00} \approx 72.27$	616
Upper Quota	88	422	1	32	73	616

Step 7: The new sum of the upper quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 88, Honolulu County 422, Kalawao County 1, Kauai 32, and Maui 73 schools.

When using Adams's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

In this apportionment, Mythbury received less than the state's lower quota. So, this apportionment is an example of a scenario in which the Adams's method violates the quota rule.

Adams's method of apportionment violates the quota rule.

So far, only Hamilton's method satisfies the quota rule, but there is one more apportionment method you should consider for Imaginaria.

VIDEO

Adams Method Apportionment Calculator

TECH CHECK

Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Adams Method apportionment calculator.

This can be a useful tool to confirm your results!

Webster's Method of Apportionment

Webster's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did or the sum of the upper quotas as Adams did, Webster used traditional rounding.

To apply Webster's method of apportionment, there are four steps we take:

1. Find the standard divisor.
2. Find each state's quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.
3. Round each state's quota to the nearest whole number and find the sum of these values.
4. If the sum of the rounded quotas equals the number of seats, the apportionment is complete. If the sum of the rounded quotas is less than the number of seats, reduce the divisor. If the sum of the rounded quotas is greater than the number of seats, increase the divisor. Return to Step 2 using the modified divisor.

When using Webster's method, just as with Jefferson's method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

Example 9 **Hawaiian School Districts**

Use the data in the table below to apportion 616 schools to Hawaiian counties. This time, use Webster's method.

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Residents under Age 19	46,310	224,230	20	16,560	38,450	325,570

Solution

To apply Webster's method of apportionment, there are four steps we take:

Step 1: The processes of finding the standard divisor and standard quota are the same in the Jefferson, Adams, and Webster's methods. As in the previous examples, the standard divisor is 528.52.

Step 2: Find each state's rounded quota and their sum:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Standard Quota	$\frac{46,310}{528.52} \approx 87.62$	$\frac{224,230}{528.52} \approx 424.26$	$\frac{20}{528.52} \approx 0.04$	$\frac{16,560}{528.52} \approx 31.33$	$\frac{38,450}{528.52} \approx 72.75$	616
Rounded Quota	88	424	1	31	73	617

Step 3: Compare the sum of the states' rounded quotas, 617, to the house size, 616. Since 617 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00.

Step 4: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 534:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{534.00} \approx 86.72$	$\frac{224,230}{534.00} \approx 419.91$	$\frac{20}{534.00} \approx 0.04$	$\frac{16,560}{534.00} \approx 31.01$	$\frac{38,450}{534.00} \approx 72.00$	616
Upper Quota	87	420	1	31	72	612

Step 5: The new sum of the rounded quotas is 4 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 530.00.

Step 6: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 530.00:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{530.00} \approx$ 87.38	$\frac{224,230}{530.00} \approx$ 423.08	$\frac{20}{530.00} \approx$ 0.04	$\frac{16,560}{530.00} \approx$ 31.25	$\frac{38,450}{530.00} \approx$ 72.55	616
Upper Quota	87	423	1	31	73	615

Step 7: The new sum of the rounded quotas is 1 unit less than 616. Larger quotas are needed. So, decrease the divisor to a value between 528.52 and 530.00. Try 529.50.

Step 8: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.50:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{529.50} \approx$ 87.46	$\frac{224,230}{529.50} \approx$ 423.48	$\frac{20}{529.50} \approx$ 0.04	$\frac{16,560}{529.50} \approx$ 31.27	$\frac{38,450}{529.50} \approx$ 72.62	616
Upper Quota	87	423	1	31	73	615

Step 9: The new sum is still only 1 unit less than 616. Larger quotas are needed, but not much larger. So, decrease the divisor to a value between 528.52 and 529.50. Try 529.30.

Step 10: Find each state's modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.30:

	Hawaii	Honolulu	Kalawao	Kauai	Maui	Total
Modified Quota	$\frac{46,310}{529.30} \approx$ 87.49	$\frac{224,230}{529.30} \approx$ 423.63	$\frac{20}{529.30} \approx$ 0.04	$\frac{16,560}{529.30} \approx$ 31.29	$\frac{38,450}{529.30} \approx$ 72.64	616
Upper Quota	87	424	1	31	73	616

Step 11: The new sum of the rounded quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 87, Honolulu County 424, Kalawao County 1, Kauai 31, and Maui 73 schools.

So far, we know that the Hamilton method satisfies the quota rule, while the Jefferson and Adams methods do not. The apportionments in the Example and Your Turn above are both scenarios in which the Webster method satisfies the quota rule. Does it always? We have a little more work to do to find out. However, one thing is clear. Not all apportionment methods have the same results. Before you make such an important decision for Imaginaria, it's important to think about the differences in the apportionments that result from these four methods. How will the differences affect the citizens of Imaginaria?

TECH CHECK

Apportionment Calculators

Check out websites such as Ms. Hearn Math for a free Webster Method apportionment calculator.

This can be a useful tool to confirm your results!

Comparing Apportionment Methods

Recall that the four apportionment methods discussed in this chapter differ in two main ways:

- Whether or not a modified divisor is used
- The type of rounding of the quotas that is used

  How might these differences affect Imaginarians? In the next two examples, we will compare the results when different apportionment methods are applied to the same scenario.

Example 10 **Hawaiian School Districts with Different Apportionment Methods**

Let's use the results from Example 4, Example 7, Example 8, and Example 9 to compare the four apportionment methods we have discussed. The following table summarizes the results of the results of the Hamilton, Jefferson, Adams and Webster methods when applied to the apportionment of 616 schools to Hawaiian counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Under 19 years old	46,310	224,230	20	16,560	38,450
Hamilton	87	424	1	31	73
Jefferson	87	425	1	31	72
Adams	88	422	1	32	73
Webster	87	424	1	31	73

1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

Solution

1. Yes, the Hamilton and Webster methods result in the same apportionment.
2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method.

They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.

3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 194 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson methods the least.

The Adams method favored the smaller states and the Jefferson method favored the larger states in the previous example, but is this the case in general?

Since the Jefferson method begins with the lower quotas, any adjustment to the quotas will be an increase. As you have seen, this is accomplished by using a modified divisor that is smaller than the standard divisor. The next example compares the impact of a decreasing divisor on the modified quotas of large states to the impact of the same size decrease on small states.

Example 11 **Effect of Decreasing Divisors on Modified Quotas**

The following table displays the effect of reducing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states and use the table answer each question.

		Modified Quotas		
State	Population	Divisor: 10,500	Divisor: 10,000	Divisor: 9,500
A	10,000	0.95	1	1.05
B	100,000	9.52	10	10.53
C	1,000,000	95.24	100	105.26

1. When the divisor decreases from 10,500 to 10,000, how many representatives are gained by each state based on the lower quota?
2. When the divisor decreases from 10,000 to 9,500, how many representatives are gained by each state based on the lower quota?
3. Which state gains the most representatives each time the divisor is decreased?

Solution

1. Since a state must have at least one seat, State A begins with 1 seat and still has one seat. State B begins with 9 seats and increases to 10 seats. State C begins with 95 seats and increases to 100 seats. So, State A gains 0, B gains 1, and C gains 5 seats.
2. State A begins with 1 and still has 1. State B begins with 10 and still has 10. State C begins with 100 and increases to 105. So, State A gains 0, State B gains 0, and State C gains 5.

3. State C, the largest state, gains the most representatives each time the divisor is decreased.

This example demonstrates that the Jefferson method is biased toward states with larger populations because the modified divisor is smaller than the standard divisor. On the other hand, the Adams's method, which begins with the upper quotas, must increase the standard divisor in order to reduce the quotas. Once again, the effect on the number of seats is greater for the larger states, but this time they are decreased. This means that the Adams's method favors states with smaller populations.

Flaws in Apportionment Methods

As we have seen, different apportionment methods can have the same results in some scenarios but different results in others. Citizens of states which receive fewer seats with a particular apportionment method will view the apportionment method as flawed and argue in favor of a different method. This inevitably creates debates regarding the use of one method over another. Methods that favor larger states are likely to be challenged by smaller states, methods that favor smaller states are likely to be challenged by larger states, and methods that violate the quota rule are likely to be challenged by states of any size depending on the circumstances.

Suppose that the State of Hawaii House of Representatives had 51 representatives, each with their own district. Imagine that redistricting were underway, and the representative districts were to be apportioned to each of five counties based on population. The following table shows the apportionment that would result from the use of the Jefferson, Adams, and Webster methods of apportionment.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Population	201,500	974,600	100	72,300	167,400
Lower Quota	7	35	0	2	6
Upper Quota	8	36	1	3	7
Jefferson	7	35	1	2	6
Adams	7	34	1	3	6
Webster	7	34	1	3	6

From the table, you can see that Hawaii, Kalawao, and Maui receive the same number of seats regardless of the method used. However, citizens of Honolulu would likely reject the Adams and Webster methods arguing that they violate the quota rule. Similarly, citizens of Kauai would probably reject the Jefferson method based on the argument that it unfairly favors the larger states. This scenario demonstrates that the Adams and Webster methods violate the quota rule, but the Jefferson method also violates the quota rule at times. **The Hamilton method is the only method that satisfies the quota rule in all scenarios.** It also consistently favors neither larger nor smaller states. Unfortunately, it can have some strange and results in certain circumstances, which you will see in the next section.

WHO KNEW?

Gerrymandering: A Subtle Way to Impact Apportionment

In addition to your choice of voting method and your choice of apportionment method, there is another important decision to make which could potentially have a huge impact on the fairness of elections in Imaginaria—the creation of electoral districts. In example above, we imagined that there were 51 state legislators in Hawaii, each representing their own district. But how did the legislators decide on the boundaries for these districts? Typically, boundaries are drawn so that each district has approximately the same number of residents, but the percentage of residents in each district with a particular political affiliation can swing the power from one group to another. When the districts are drawn to impact the power of a political party, ethnic or racial group, or other group, this is called gerrymandering. For example, districts can be drawn so that a particular group is spread thinly across districts, increasing the likelihood that they will not have strong representation.

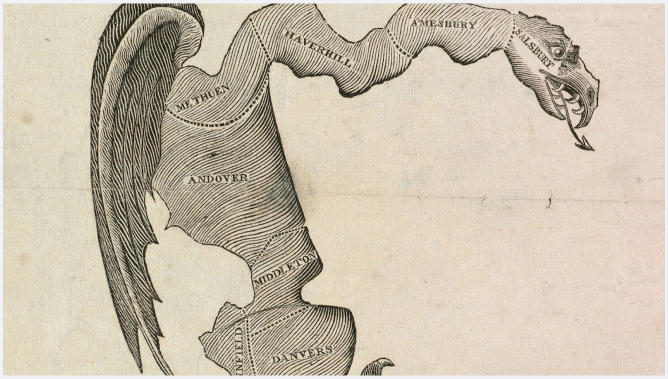

Figure 17: This cartoon map conveys the idea that the drawing of the map may impact election outcomes.

"The Gerry-mander" first appeared in this cartoon-map in the Boston Gazette, March 26, 1812, and was soon reproduced in several other Massachusetts newspapers in response to election district changes initiated by governor Eldridge Gerry. Note that while the practice is named after him, Gerry was not the first to employ it.

PEOPLE IN MATHEMATICS

Jonathan Mattingly

Jonathan Mattingly is a mathematician who was featured in a *Nature* article titled "The Mathematicians Who Want to Save Democracy". Mattingly is a mathematician at Duke University in North Carolina and he runs election simulations based on alternate versions of electoral districts in order to analyze the effects of gerrymandering. He has even been asked to testify as an expert witness in court. Mattingly and other mathematicians who are working on the problem will potentially have an impact on the redistricting that will occur

> as a result of the 2020 census. (Carrie Arnold, “The Mathematicians Who Want to Save Democracy,” *Nature 546*, 200–202, 2017.)

Key Terms

- apportionment problem
- lower quota
- upper quota

Key Concepts

- Hamilton’s method of apportionment uses the standard divisor and standard lower quotas, and it distributes any remaining seats based on the size of the fractional parts of the standard lower quota. Hamilton’s method satisfies the quota rule and favors neither larger nor smaller states.
- Jefferson’s method of apportionment uses a modified divisor that is adjusted so that the modified lower quotas sum to the house size. Jefferson’s method violates the quota rule and favors larger states.
- Adams’s method of apportionment uses a modified divisor that is adjusted so that the modified upper quotas, sum to the house size. Adams’s method violates the quota rule and favors smaller states.
- Webster’s method of apportionment uses a modified divisor that is adjusted so that the modified state quotas, rounded using traditional rounding, sum to the house size. Webster’s method violates the quota rule but favors neither larger nor smaller states.

Videos

- Hamilton Method of Apportionment
- Jefferson Apportionment Method
- Adams Method Apportionment Calculator

11.5 Fairness in Apportionment Methods

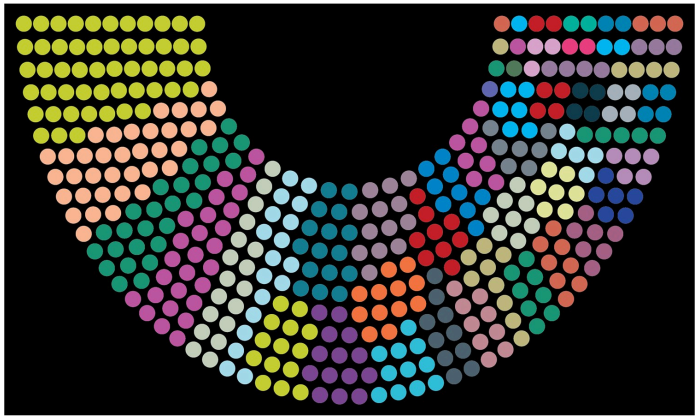

Figure 18: In this seating chart for the House of Representatives, each color indicates representatives from a particular state.

Learning Objectives

After completing this section, you should be able to:

1. Describe and illustrate the Alabama paradox.
2. Describe and illustrate the population paradox.
3. Describe and illustrate the new-states paradox.
4. Identify ways to promote fairness in apportionment methods.

Apportionment Paradoxes

The citizens of Imaginaria will want the apportionment method to be as fair as possible. There are certain characteristics that they would reasonably expect from a fair apportionment.

- If the house size is increased, the state quotas should all increase or remain the same but never decrease.
- If one state's population is growing more rapidly than another state's population, the faster growing state should not lose a seat while a slower growing state maintains or gains a seat.
- If there is a fixed number of seats, adding a new state should not cause an existing state to gain seats while others lose them.

However, apportionment methods are known to contradict these expectations. Before you decide on the right apportionment for Imaginarians, let's explore the **apportionment paradox**, a situation that occurs when an apportionment method produces results that seem to contradict reasonable expectations of fairness.

There is a lot that the founders of Imaginaria can learn from U.S. history. The constitution of the United States requires that the seats in the House of Representatives be apportioned according to the results of the census that occurs every decade, but the number of seats and the apportionment method is not stipulated. Over the years, several different apportionment methods and house sizes have been used and scrutinized for fairness. This scrutiny has led to the discovery of several of these apportionment paradoxes.

The Alabama Paradox

At the time of the 1880 U.S. Census, the Hamilton method of apportionment had replaced the Jefferson method. The number of seats in the House of Representatives was not fixed. To achieve the fairest apportionment possible, the house sizes were chosen so that the Hamilton and Webster methods would result in the same apportionment. The chief clerk of the Census Bureau calculated the apportionments for house sizes between 275 and 350. There was a surprising result that became known as the **Alabama paradox**, which is said to occur when an increase in house size reduces a state's quota. Alabama would receive eight seats with a house size of 299, but only receive seven seats if the house size increased to 300. (Michael J. Caulfield (Gannon University), "Apportioning Representatives in the United States Congress - Paradoxes of Apportionment," *Convergence* (November 2010), DOI:10.4169/loci003163)

Example 1 **The 1880 Alabama Quota**

The 1880 census recorded the population of Alabama as 1,513,401 and that of the U.S. as 62,979,766.

1. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 299.
2. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 300.
3. Did the standard quota increase or decrease when the house size increased?
4. Consider the Hamilton method of apportionment. Explain how Alabama's final quota could be smaller with a larger standard quota.

Solution

1. The standard divisor is $62,979,766 \div 299 = 210$, resulting in 634.6689 citizens per seat. The standard quota for Alabama is $1,513,401 \div 210,634.7 = 7.1850$ seats.
2. The standard divisor is $62,979,766 \div 300 = 209$, resulting in 932.5533 citizens per seat. The standard quota for Alabama is $1,513,401 \div 209,932.6 = 7.2090$ seats.
3. The standard quota increased.
4. In each case, the state would receive the lower quota of 7 and then be awarded one more seat if the fractional part of the standard quota were high enough relative to the fractional parts of the other states' standard quotas. When the house size was 299, Alabama received one of the remaining seats after the lower quotas were distributed. When the house size was 300, Alabama did not receive one of the remaining seats after the lower quotas were distributed. It must have been the case that either the fractional part 0.2090 ranked lower amongst the other fractional parts of the state quotas than the fractional part 0.1850 did, or there were fewer remaining seats, or both.

After the 1900 census, the Census Bureau again calculated the apportionment based on various house sizes. It was determined that Colorado would receive three seats with a house size of 356, but only two seats with a house size of 357.

Example 2 **Hamilton's Method and the Alabama Paradox**

Suppose that States A and B each have a population of 6, while State C has a population of 2.

1. Use the Hamilton method to apportion 10 seats.
2. Use the Hamilton method to apportion 11 seats.
3. Does this example demonstrate the Alabama paradox? If so, how?

Solution

1. **Step 1:** The total population is 14. The standard divisor is $14 \div 10 = 1.4$ individuals per seat.

Step 2: The states' standard quotas are as follows: A $6 \div 1.4 \approx 4.29$, B $6 \div 1.4 \approx 4.29$, and C $2 \div 1.4 \approx 1.43$.

Step 3: The states' lower quotas are as follows: A 4, B 4, and C 1.

Step 4: The sum of the lower quotas is 9, which means there is one seat remaining to be apportioned. State C has the highest fractional part and receives the additional seat.

Step 5: The final apportionment is as follows: A 4, B 4, and C 2, which sums to 10.

1. **Step 1:** The total population is 14. The standard divisor is $14 \div 11 \approx 1.2727$ individuals per seat.

Step 2: The states' standard quotas are: A $6 \div 1.2727 \approx 4.71$, B $6 \div 1.2727 \approx 4.71$, and C $2 \div 1.2727 \approx 1.57$.

Step 3: The states' lower quotas are: A 4, B 4, and C 1.

Step 4: The sum of the lower quotas is 9, which means there are two seats remaining to be apportioned. A and B have the highest fractional parts and receive the additional seats.

Step 5: The final apportionment is: A 5, B 5, and C 1.

1. Yes, this demonstrates the Alabama paradox because State C receives two seats if the house size is 10, but only one seat if the house size is 11.

The Population Paradox

It is important for the founders of Imaginaria to keep in mind that the populations of states change as time passes. Some populations grow and some shrink. Some populations increase by a large amount while others increase by a small amount. These changes may necessitate a **reapportionment** of seats, or the recalculation of state quotas due to a change in population. It would be reasonable for Imaginarians to expect that the state with a population that has grown more than others will gain a seat before the other states. Once again, this is not always the case with the Hamilton method of apportionment. The **population growth rate** of a state is the ratio of the change in the population to the original size of the population, often expressed as a percentage. This value is positive if the population is increasing and negative if the population is decreasing. The **population paradox** occurs when a state with an increasing population loses a seat while a state with a decreasing population either retains or gains seats. More generally, the population paradox occurs when a state with a higher population growth rate loses seats while a state with a lower population growth rate retains or gains seats.

Notice that the population paradox definitions has two parts. If either part applies, then the population paradox has occurred. The first part of the definition only applies when one state has a decreasing population. The second part of the definition applies in all situations, whether there is a state with a decreasing population or not. It will be easier to identify situations that involve a decreasing population. The other situations requires the calculation of a growth rate. The reason that we don't have to calculate a growth rate when one state has a decreasing population and the other has an increasing population is that increasing population has a positive growth rate which is always greater than the negative growth rate of a decreasing population.

CHECKPOINT

A state must lose a seat in order for the population paradox to apply. It is not enough for a state with a lower growth rate to gain a seat while a state with a higher growth rate retains the same number of seats.

Example 3 **Apportionment of Respirators to Hospitals**

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The Hamilton method is used to allocate the respirators in 2020, then to reallocate based on new capacities in 2021. The results are shown in the table below. How does this demonstrate the population paradox?

Hospital	Capacity in 2020	Respirators in 2020	Change in Capacity	Growth Rate $= \frac{\text{Change in Capacity}}{\text{Capacity in 2020}}$	Capacity in 2021	Respirators in 2021
A	825	9	57	$\frac{57}{825} \approx 0.0691 = 6.91\%$	882	9
B	613	7	13	$\frac{13}{613} \approx 0.0212 = 2.12\%$	626	6
C	239	2	3	$\frac{3}{239} = 0.0126 = 1.26\%$	242	3

Solution

Hospital B lost a respirator while hospital C gained one, even though hospital B had a higher growth rate than hospital C.

FORMULA

The growth rate of a population can be calculated by subtracting the previous population size from the current population size, and then dividing the difference by the previous population size.

$$\text{population growth rate} = \frac{\text{current population size} - \text{previous population size}}{\text{previous population size}}$$

CHECKPOINT

Make sure to calculate the subtraction before the division. If you are entering the values in a calculator, it is helpful to put parentheses around the subtracted terms.

Example 4 **The Congress of Costaguana**

The country of Costaguana has three states: Azuera with a population of 894,000; Punta Mala with a population of 696,000; and Esmeralda with a population of 215,000. There are 38 seats in the Congress of Costaguana. The apportionment of the seats is determined by Hamilton's method to be: 19 for Azuera, 15 for Punta Mala, and 4 for Esmerelda. A census reveals that the population has grown and the seats must be reapportioned. If Azuera now has 953,000 residents, Punta Mala now has 706,000 residents, and Esmerelda now has 218,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox?

Solution

The Hamilton reapportionment is: 19 for Azuera, 14 for Punta Mala, and 5 for Esmerelda. This is an example of the population paradox because Punta Mala lost a seat to Esmerelda, even though Punta Mala's population grew by 1.44 percent while Esmerelda's only grew by 1.40 percent.

The New-States Paradox

As a founder of Imaginaria, you might also consider the possibility that Imaginaria could annex nearby lands and increase the number of states. This occurred several times in the United States such as when Oklahoma became a state in 1907. The House size was increased from 386 to 391 to accommodate Oklahoma's quota of five seats. When the seats were reapportioned using Hamilton's method, New York lost a seat to Maine despite the fact that their populations had not changed. This is an example of the **new-state paradox**, which occurs when the addition of a new state is accompanied by an increase in seats to maintain the standard ratio of population to seats, but one of the existing states loses a seat in the resulting reapportionment.

Example 5 **New State of Oscuridad**

The country of San Lorenzo has grown from two states to three. The house size of the congress has been increased by eight and the seats have been reapportioned to accommodate the new state of Oscuridad. The constitution mandates the use of the Hamilton method of apportionment. Use this information and the following table to answer the questions.

State	Population (in hundreds)	Original Apportionment	Reapportionment
Clara	7,100	39	40
Velasco	9,080	51	50
Oscuridad	1,500	Not Applicable	8

1. What was the original house size?
2. What is the new house size?
3. How is this reapportionment an example of the new-states paradox?

Solution

1. $39 + 51 = 90$
2. $40 + 50 + 8 = 98$
3. The original state of Velasco lost a seat to the original state of Clara when the new state of Oscuridad was added.

Example 6 **The Growing Country of Gulliversia**

The country of Gulliversia has two states: Lilliput with a population of 700,000 and Brobdingnag with a population of 937,000. The constitution of Gulliversia requires that the 90 congressional seats be apportioned by Hamilton's method. Lilliput has received 38 seats while Brobdingnag has received 52 seats. Recently, the island of Houyhnhnmsland with a population of 170,000 has joined the union, becoming a state of Gulliversia. When Houyhnhnmsland is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton's method to reapportion the 99 seats to the three states. How is the resulting apportionment an example of the new-states paradox?

Solution

The reapportionment gives 39 seats to Lilliput, 51 seats to Brobdingnag, and 9 seats to Houyhnhnmsland. This is an example of the new-states paradox because the original state of Brobdingnag lost a seat to the original state of Lilliput when the new state was added to the union.

When a new state is added, it is necessary to determine the amount that the house size must be increased to retain the original ratio of population to seats, in other words to keep the original standard divisor. To calculate the new house size, divide the new population by the original standard divisor, and round to the nearest whole number.

FORMULA

$$\text{New House Size} = \frac{\text{New Population}}{\text{Original Standard Divisor}} \text{ rounded to the nearest whole number.}$$

Example 7 **Oklahoma Joins the Union**

Oklahoma was admitted as the 46th state on November 16, 1907. Before Oklahoma joined the union, the U.S. population was approximately 75,030,000 and the House of Representatives had 386 seats. The new state had a population of approximately 970,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats Oklahoma should receive.

Solution

Step 1: Original Standard Divisor $\approx \frac{75,030,000}{386} \approx 194,400$

Step 2: New Population $\approx 75,030,000 + 970,000 = 76,000,000$

Step 3: New House Size $\approx \frac{76{,}000{,}000}{194,400} \approx 391$

Step 4: There are $391 - 386 = 5$ new seats to be apportioned to Oklahoma.

The Search for the Perfect Apportionment Method

The ideal apportionment method would simultaneously satisfy the following four fairness criteria.

- Satisfy the quota rule
- Avoid the Alabama paradox
- Avoid the population paradox
- Avoid the new-states paradox

We have seen that the Hamilton method allows the Alabama paradox, the population paradox, and the new-states paradox in some apportionment scenarios. Let's explore the results of the other methods of apportionment we have discussed in some of the same scenarios.

Example 8 **Orange Grove and the New-States Paradox**

The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents and The Villages with 10,746 residents. A council with 100 members supervises the municipality's operations. The council votes to annex an unincorporated subdivision called The Lakes with a population of 630. They plan to increase the size of the council to maintain the ratio of seats to residents such that the new council will have 100 seats plus the number of seats given to The Lakes. Use each of the following apportionment methods and indicated number of additional seats to find the original and new apportionment and determine whether the new-state paradox occurs.

1. Jefferson's method with five additional seats.

2. Adams's method with six additional seats.
3. Webster's method with five additional seats

Solution

1. Using a modified divisor of 119, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.
2. Using a modified divisor of 121, the original apportionment would have been: The Oaks 11 and The Villages 89. Using a modified divisor of 121, the new apportionment would be: The Oaks 11, The Villages 89, and The Lakes 6. The new-state paradox does not occur.
3. Using the standard divisor of 120, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119.5, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.

We have seen in our examples that neither the population paradox nor the new-states paradox occurred when using the Jefferson, Adams, and Webster methods. It turns out that, although all three of these divisor methods violate the quota rule, none of them ever causes the population paradox, new-states paradox, or even the Alabama paradox. On the other hand, the Hamilton method satisfies the quota rule, but will cause the population paradox, the new-states paradox, and the Alabama paradox in some scenarios.

In 1983, mathematicians Michel Balinski and Peyton Young proved that no method of apportionment can simultaneously satisfy all four fairness criteria.

There are other apportionment methods that satisfy different subsets of these fairness criteria. For example, the mathematicians, Balinski and Young who proved the Balinski-Young Impossibility Theorem created a method that both satisfies the quota rule and is free of the Alabama paradox. (Balinski, Michel L.; Young, H. Peyton (November 1974). "A New Method for Congressional Apportionment." *Proceedings of the National Academy of Sciences*. 71 (11): 4602–4606.) However, no method may always follow the quota rule and simultaneously be free of the population paradox. (Balinski, Michel L.; Young, H. Peyton (September 1980). "The Theory of Apportionment" (PDF). Working Papers. International Institute for Applied Systems Analysis. WP-80-131.)

So, as you and your fellow founders of Imaginaria make the important decision about the right apportionment method for Imaginaria, do not look for a perfect apportionment method. Instead, look for an apportionment method that best meets the needs and concerns of Imaginarians.

Key Terms

- apportionment paradox

- Alabama paradox
- reapportionment
- population growth rate
- population paradox
- new-state paradox

Key Concepts

- Several surprising outcomes can occur when apportioning seats that voters may find unfair: Alabama paradox, population paradox, and new-state paradox.
- Apportionment methods are susceptible to apportionment paradoxes and may violate the quota rule.
- The Balinsky-Young Impossibility Theorem indicates that no apportionment can satisfy all fairness criteria.

Formulas

$$\text{population growth rate} = \frac{\text{current population size} - \text{previous population size}}{\text{previous population size}}$$

$$\text{New House Size} = \frac{\text{New Population}}{\text{Original Standard Divisor}} \text{ rounded to the nearest whole number.}$$

Projects

The First Census: Challenges of Collecting Population Data

As you and your fellow founders write the Imaginarian constitution, you must also design systems to collect accurate information about the population of Imaginaria. Why is this important and what are the challenges you will face? Let's find out! Research and answer each question. (Questions adapted from "Authorizing the First Census–The Significance of Population Data," Census.gov, United States Census Bureau)

1. The First Congress laid out a plan for collecting this data in Chapter II, *An Act Providing for the Enumeration of the Inhabitants of the United States*, which was approved March 1, 1790. What group did Congress select to carry out the first enumeration? Why did they choose this group? What might be the advantages and disadvantages to this approach?
2. Choose at least two challenges faced by the U.S. government during the first enumeration and explain how the information gathered might have helped address them.
3. President James Madison was a U.S. Representative who participated in the census debate in the first congress. What were the main points of his remarks and how were they relevant to the overall debate about the first enumeration?
4. What issues do you think the U.S. Census Bureau encounters today as it continues to collect and process data about the U.S. population that might be significant to you and the other founders of Imaginaria?

The Party System: How Many Political Parties Are Enough?

The electoral system of Imaginaria will likely involve multiple political parties. The way these parties interact with the system may be determined by the founders of the new democracy. Let's explore the ways in which political parties interact with the governments of various democracies by researching and answering each of the following questions.

1. What concerns did the founders of the United States have about political parties? Have any of their concerns become a reality? Were political parties addressed in the U.S. Constitution? How did they become such a large part of politics in the U.S.? As a founder of Imaginaria, would you address political parties in your constitution?
2. What is frontloading? Does our current system of frontloading impact fair representation? Why do two small, racially homogeneous states hold their primaries first? Do you think this impacts the final results?
3. How does the interaction political parties and the electoral system in the United States differ from that of other countries? Give at least three specific examples. Of these three, which would you be most likely to use as a model for Imaginaria?

Technology and Voting: How Has the Digital Age Impacted Elections?

Unlike most other countries, Imaginaria will be founded in the digital age. Let's explore the impact this might have on how you choose to set up your electoral system. Research and answer each question.

1. Participation in an electoral system if very important. In what ways does the Internet positively affect participation? In what ways does it negatively affect participation. What roll, if any, should government of Imaginaria play in tempering or promoting the Impact of the Internet?
2. Find at least three examples of governments that utilize Internet voting around the world. What concerns have slowed the spread of this technology? What are the advantages and disadvantages of Internet voting? Would you be in favor of Internet voting for Imaginaria? Why or why not?