Chapter 10: Geometry

Figure 1: *The School of Athens* by the Renaissance artist Raphael, painted between 1509 and 1511, depicts some of the greatest minds of ancient times.

The painting *The School of Athens* presents great figures in history such as Plato, Aristotle, Socrates, Euclid, Archimedes, and Pythagoras. Other scientists are also represented in the painting.

To the ancient Greeks, the study of mathematics meant the study of geometry above all other subjects. The Greeks looked for the beauty in geometry and did not allow their geometrical constructions to be "polluted" by the use of anything as practical as a ruler. They permitted the use of only two tools—a compass for drawing circles and arcs, and an unmarked straightedge to draw line segments. They would mark off units as needed. However, they never could be sure of what the units meant. For instance, how long is an inch? These mathematicians defined many concepts.

The Greeks absorbed much from the Egyptians and the Babylonians (around 3000 BCE), including knowledge about congruence and similarity, area and volume, angles and triangles, and made it their task to introduce proofs for everything they learned. All of this historical wisdom culminated with Euclid in 300 BCE.

Euclid (325–265 BC) is known as the father of geometry, and his most famous work is the 13-volume collection known as *The Elements*, which are said to be "the most studied books apart from the Bible." Euclid brought together everything offered by the Babylonians, the Egyptians, and the more refined contributions by the Greeks, and set out, successfully, to organize and prove these concepts as he methodically developed formal theorems.

This chapter begins with a discussion of the most basic geometric tools: the point, the line, and the plane. All other topics flow from there. Throughout the eight sections, we will talk about how to determine angle measurement and learn how to recognize properties of special angles, such as right angles and supplementary angles. We will look at the relationship of angles formed by a transversal, a line running through a set of parallel lines. We will explore the concepts of area and perimeter, surface area and volume, and transformational geometry as used in the patterns and rigid motions of tessellations. Finally, we will introduce right-angle trigonometry and explore the Pythagorean Theorem.

10.1 Points, Lines, and Planes

Figure 2: The lower right-hand corner of *The School of Athens* depicts a figure representing Euclid illustrating to students how to use a compass on a small chalkboard.

Learning Objectives

After completing this section, you should be able to:

1. Identify and describe points, lines, and planes.
2. Express points and lines using proper notation.
3. Determine union and intersection of sets.

In this section, we will begin our exploration of geometry by looking at the basic definitions as defined by Euclid. These definitions form the foundation of the geometric theories that are applied in everyday life.

In *The Elements*, Euclid summarized the geometric principles discovered earlier and created an axiomatic system, a system composed of postulates. A *postulate* is another term for axiom, which is a statement that is accepted as truth without the need for proof or verification. There were no formal geometric definitions before Euclid, and when terms could not be defined, they could be described. In order to write his postulates, Euclid had to describe the terms he needed and he called the descriptions "definitions." Ultimately, we will work with theorems, which are statements that have been proved and can be proved.

Points and Lines

The first definition Euclid wrote was that of a point. He defined a point as "that which has no part." It was later expanded to "an indivisible location which has no width, length, or breadth." Here are the first two of the five postulates, as they are applicable to this first topic:

1. **Postulate 1**: A straight line segment can be drawn joining any two points.
2. **Postulate 2**: Any straight line segment can be extended indefinitely in a straight line.

   Before we go further, we will define some of the symbols used in geometry in :

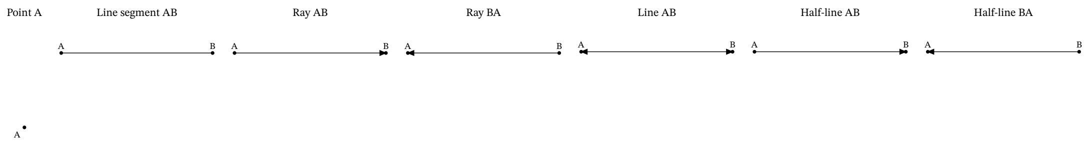


Figure 3: Basic Geometric Symbols for Points and Lines

From , we see the variations in lines, such as line segments, rays, or half-lines. What is consistent is that two collinear points (points that lie on the same line) are required to form a line. Notice that a line segment is defined by its two endpoints showing that there is a definite beginning and end to a line segment. A ray is defined by two points on the line; the first point is where the ray begins, and the second point gives the line direction. A half-line is defined by two points, one where the line starts and the other to give direction, but an open circle at the starting point indicates that the starting point is not part of the half-line. A regular line is defined by any two points on the line and extends infinitely in both directions. Regular lines are typically drawn with arrows on each end.

Example 1 **Defining Lines**

For the following exercises, use this line .

1. Define $\overline{DE}$.
2. Define F.
3. Define $\overleftrightarrow{DF}$.
4. Define $\overline{EF}$.

Solution

1. The symbol $\overline{DE}$, two letters with a straight line above, refers to the line segment that starts at point D and ends at point E.
2. The letter F alone refers to point F.
3. The symbol $\overleftrightarrow{DF}$, two letters with a line above containing arrows on both ends, refers to the line that extends infinitely in both directions and contains the points D and F.
4. The symbol $\overline{EF}$, two letters with a straight line above, refers to the line segment that starts at point E and ends at point F.

There are numerous applications of line segments in daily life. For example, airlines working out routes between cities, where each city's airport is a point, and the points are connected by line segments. Another example is a city map. Think about the intersection of roads, such that the center of each intersection is a point, and the points are connected by line segments representing the roads.

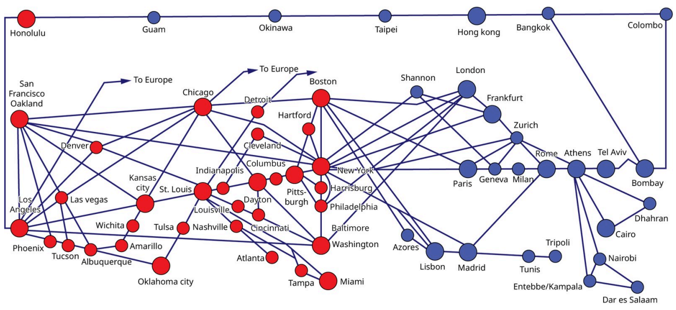


Figure 5: Air Line Routes

Example 2 **Determining the Best Route**

View the street map as a series of line segments from point to point. For example, we have vertical line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ on the right. On the left side of the map, we have vertical line segments $\overline{HI}$, $\overline{FG}$. The horizontal line segments are $\overline{HA}$, $\overline{EI}$, $\overline{IE}$, $\overline{EB}$, $\overline{FC}$, $\overline{CF}$, and $\overline{GB}$. There are two diagonal line segments, $\overline{AE}$ and $\overline{EF}$. Assume that each location is on a corner and that you live next door to the library.

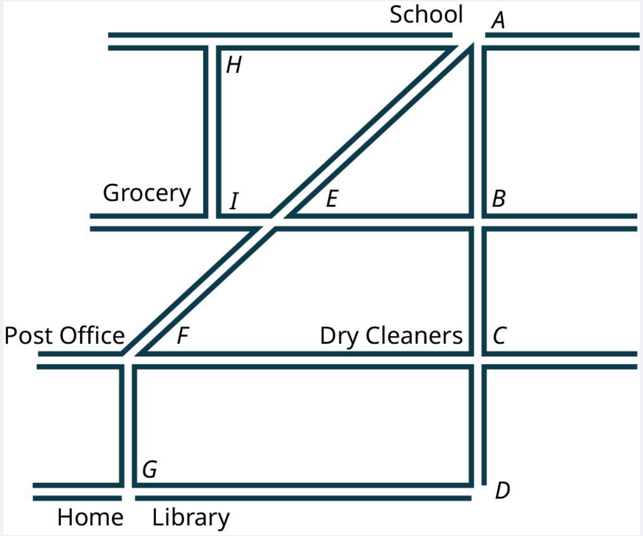


Figure 6: Street Map

1. Let's say that you want to stop at the grocery store on your way home from school. Come up with three routes you might take to do your errand and then go home. In other words, name the three ways by the line segments in the order you would walk, and which way do you think would be the most efficient route?
2. How about stopping at the library after school? Name four ways you might travel to the library and which way do you think is the most efficient?

3. Suppose you need to go to the post office and the dry cleaners on your way home from school. Name three ways you might walk to do your errands and end up at home. Which way do you think is the most efficient way to walk, get your errands done, and go home?

Solution

1. From school to the grocery store and home: first way $\overline{AH}, \overline{HI}, \overline{IE}, \overline{EF}, \overline{FG}$ second way $\overline{AB}, \overline{BE}, \overline{EI}, \overline{IE}, \overline{EF}, \overline{FG}$ third way $\overline{AE}, \overline{EI}, \overline{IE}, \overline{EF}, \overline{FG}$. It seems that the third way is the most efficient way.
2. From school to the library: first way $\overline{AE}, \overline{EF}, \overline{FG}$ second way $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DG}$ third way $\overline{AB}, \overline{BE}, \overline{EF}, \overline{FG}$ fourth way $\overline{AB}, \overline{BC}, \overline{CF}, \overline{FG}$. The first way should be the most efficient way.
3. From school to the post office or dry cleaners to home: first way $\overline{AE}, \overline{EF}, \overline{FC}, \overline{CD}, \overline{DG}$ second way $\overline{AB}, \overline{BE}, \overline{EF}, \overline{FC}, \overline{CD}, \overline{DG}$ third way $\overline{AB}, \overline{BC}, \overline{CF}, \overline{FG}$. The third way would be the most efficient way.

Parallel Lines

Parallel lines are lines that lie in the same plane and move in the same direction, but never intersect. To indicate that the line l_1 and the line l_2 are parallel we often use the symbol $l_1 \parallel l_2$. The distance d between parallel lines remains constant as the lines extend infinitely in both directions.

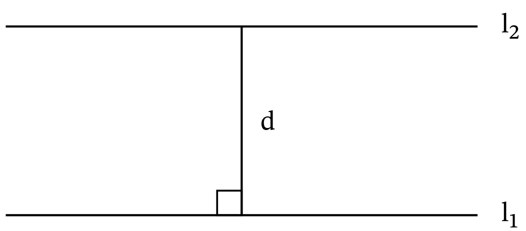


Figure 7: Parallel Lines

Perpendicular Lines

Two lines that intersect at a 90° angle are **perpendicular lines** and are symbolized by ⊥. If l_1 and l_2 are perpendicular, we write $l_1 \perp l_2$. When two lines form a right angle, a 90° angle, we symbolize it with a little square □.

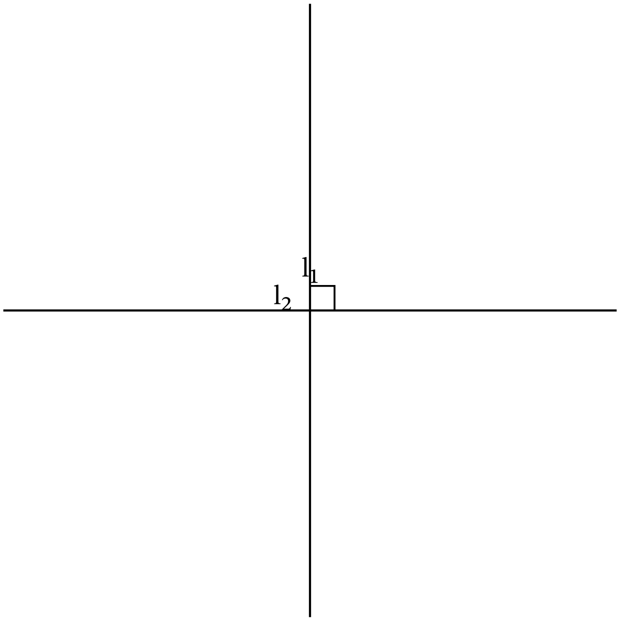
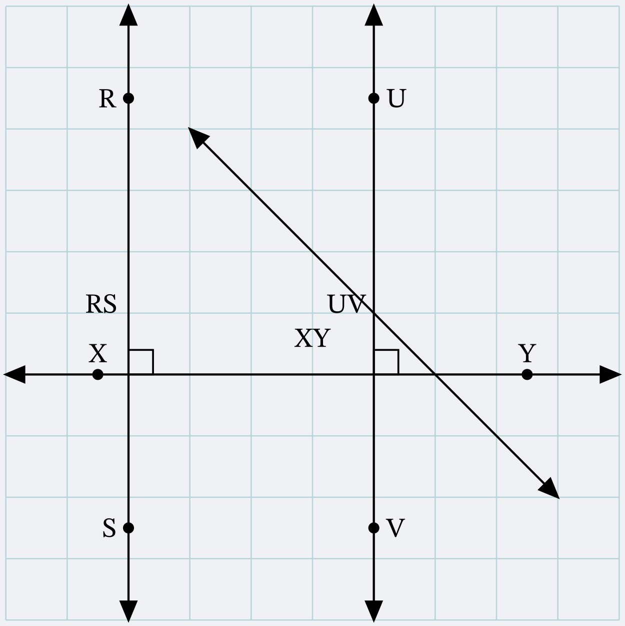


Figure 8: Perpendicular Lines

Example 3 **Identifying Parallel and Perpendicular Lines**

Identify the sets of parallel and perpendicular lines in .



Solution

Drawing these lines on a grid is the best way to distinguish which pairs of lines are parallel and which are perpendicular. Because they are on a grid, we assume all lines are equally spaced across the grid horizontally and vertically. The grid also tells us that the vertical lines are parallel and the horizontal lines are parallel. Additionally, all intersections form a 90° angle. Therefore, we can safely say the following:

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, the line containing the points A and B is parallel to the line containing the points C and D.

$\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$, the line containing the points E and F is parallel to the line containing the points G and H.

$\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$, the line containing the points A and B is perpendicular to the line containing the points E and F. We know this because both lines trace grid lines, and intersecting grid lines are perpendicular.

We can also state that $\overleftrightarrow{AB} \perp \overleftrightarrow{GH}$ the line containing the points A and B is perpendicular to the line containing the points G and H because both lines trace grid lines, which are perpendicular by definition.

We also have $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$ the line containing the points C and D is perpendicular to the line containing the points E and F because both lines trace grid lines, which are perpendicular by definition.

Finally, we see that $\overleftrightarrow{CD} \perp \overleftrightarrow{GH}$ the line containing the points C and D is perpendicular to the line containing the points G and H because both lines trace grid lines, which are perpendicular by definition.

Defining Union and Intersection of Sets

Union and intersection of sets is a topic from set theory that is often associated with points and lines. So, it seems appropriate to introduce a mini-version of set theory here. First, a **set** is a collection of objects joined by some common criteria. We usually name sets with capital letters. For example, the set of odd integers between 0 and 10 looks like this: $A = \{1, 3, 5, 7, 9\}$. When it involves sets of lines, line segments, or points, we are usually referring to the **union** or **intersection** of set.

The union of two or more sets contains all the elements in either one of the sets or elements in all the sets referenced, and is written by placing this symbol $\cup$ in between each of the sets. For example, let set $A = \{1, 2, 3\}$, and let set $B = \{4, 5, 6\}$. Then, the union of sets A and B is $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

The intersection of two or more sets contains only the elements that are common to each set, and we place this symbol $\cap$ in between each of the sets referenced. For example, let's say that set $A = \{1, 3, 5\}$, and let set $B = \{5, 7, 9\}$. Then, the intersection of sets A and B is $A \cap B = \{5\}$.

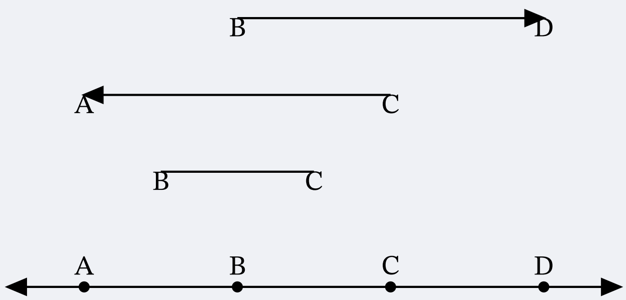
Example 4 **Defining Union and Intersection of Sets**

Use the line for the following exercises. Draw each answer over the main drawing.

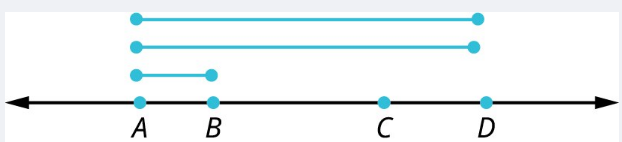
1. Find $\overrightarrow{BD} \cap \overleftarrow{CA}$.
2. Find $\overline{AB} \cup \overline{AD}$.
3. Find $\overleftrightarrow{AD} \cup \overline{BC}$.
4. Find $\overleftrightarrow{AD} \cap \overline{BC}$.
5. Find $\overleftarrow{BA} \cap \overline{CD}$.

Solution

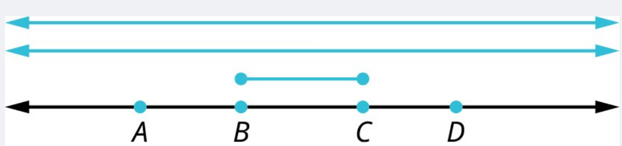
1. Find $\overrightarrow{BD} \cap \overleftarrow{CA}$. This is the intersection of the ray $\overrightarrow{BD}$ and the ray $\overleftarrow{CA}$. Intersection includes only the elements that are common to both lines. For this intersection, only the line segment $\overline{BC}$ is common to both rays. Thus, $\overrightarrow{BD} \cap \overleftarrow{CA} = \overline{BC}$.



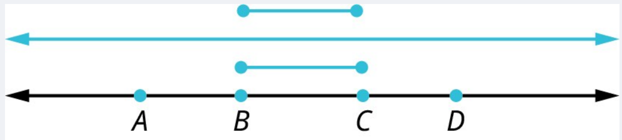
2. Find $\overline{AB} \cup \overline{AD}$. The problem is asking for the union of two line segments, $\overline{AB}$ and $\overline{AD}$. Union includes all elements in the first line and all elements in the second line. Since $\overline{AB}$ is part of $\overline{AD}$, $\overline{AB} \cup \overline{AD} = \overline{AD}$.



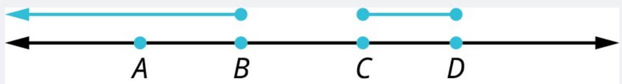
3. Find $\overleftrightarrow{AD} \cup \overline{BC}$. This is the union of the line $\overleftrightarrow{AD}$ with the line segment $\overline{BC}$. As the line segment $\overline{BC}$ is included on the line $\overleftrightarrow{AD}$, then the union of these two lines equals the line $\overleftrightarrow{AD}$.



4. Find $\overleftrightarrow{AD} \cap \overline{BC}$. The intersection of the line $\overleftrightarrow{AD}$ with line segment $\overline{BC}$ is the set of elements common to both lines. In this case, the only element in common is the line segment $\overline{BC}$.



5. Find $\overleftarrow{AB} \cap \overline{CD}$. This is the intersection of the ray $\overleftarrow{AB}$ and the line segment $\overline{CD}$. Intersection includes elements common to both lines. There are no elements in common. The intersection yields the empty set, as shown. Therefore, $\overleftarrow{AB} \cap \overline{CD} = \varnothing$.



Planes

A **plane,** as defined by Euclid, is a "surface which lies evenly with the straight lines on itself." A plane is a two-dimensional surface with infinite length and width, and no thickness. We also identify a plane by three noncollinear points, or points that do not lie on the same line. Think of a piece of paper, but one that has infinite length, infinite width, and no thickness. However, not all planes must extend infinitely. Sometimes a plane has a limited area.

We usually label planes with a single capital letter, such as Plane P, as shown in , or by all points that determine the edges of a plane. In the following figure, Plane P contains points A and B, which are on the same line, and point C, which is not on that line. By definition, P is a plane. We can move laterally in any direction on a plane.

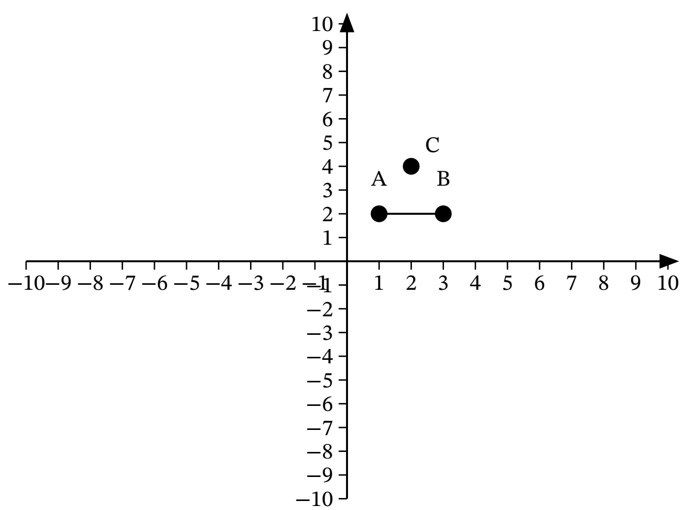


Figure 16: Plane P

One way to think of a plane is the Cartesian coordinate system with the x-axis marked off in horizontal units, and y-axis marked off in vertical units. In the Cartesian plane, we can identify the different types of lines as they are positioned in the system, as well as their locations.

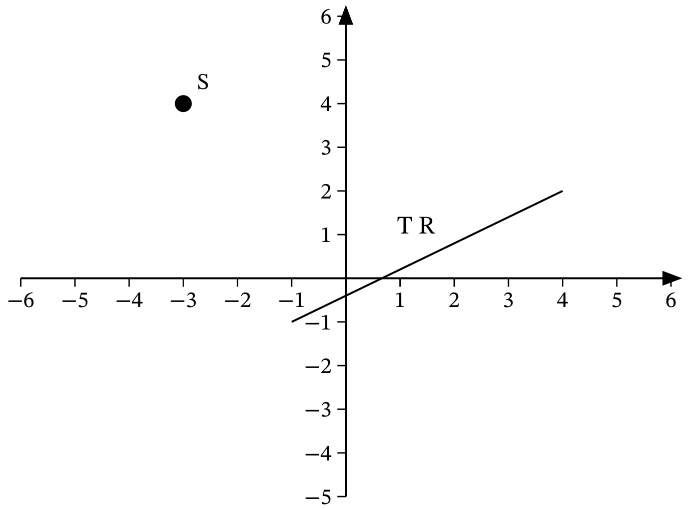


Figure 17: Cartesian Coordinate Plane

This plane contains points S, T, and R. Points T and R are colinear and form a line segment. Point S is not on that line segment. Therefore, this represents a plane.

To give the location of a point on the Cartesian plane, remember that the first number in the ordered pair is the horizontal move and the second number is the vertical move. Point R is located at $(4, 2)$; point S is located at $(-3, 4)$; and point T is located at $(-1, -1)$. We can also identify the line segment as $\overline{TR}$.

Two other concepts to note: Parallel planes do not intersect and the intersection of two planes is a straight line. The equation of that line of intersection is left to a study of three-dimensional space.

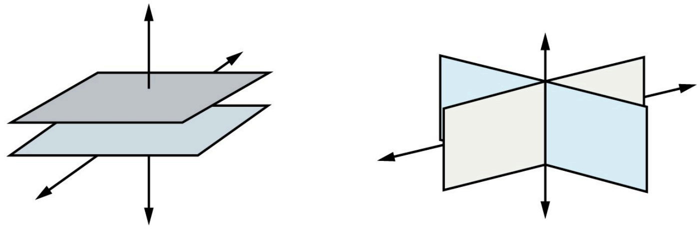


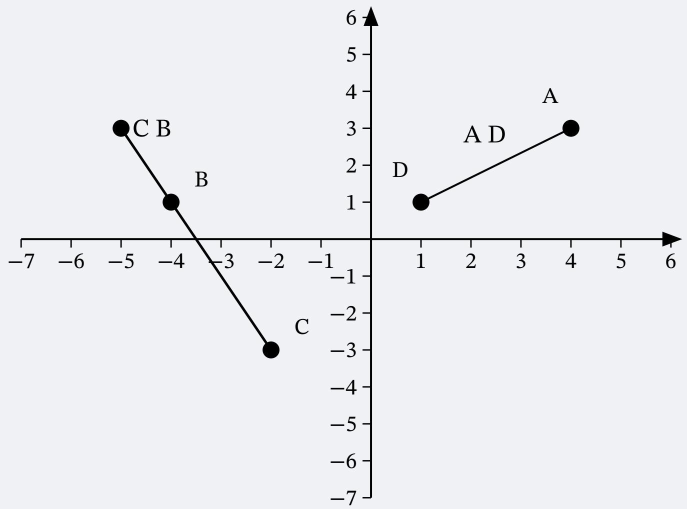
Figure 18: Parallel and Intersecting Planes

To summarize, some of the properties of planes include:

- Three points including at least one noncollinear point determine a plane.
- A line and a point not on the line determine a plane.
- The intersection of two distinct planes is a straight line.

Example 5 **Identifying a Plane**

For the following exercises, refer to



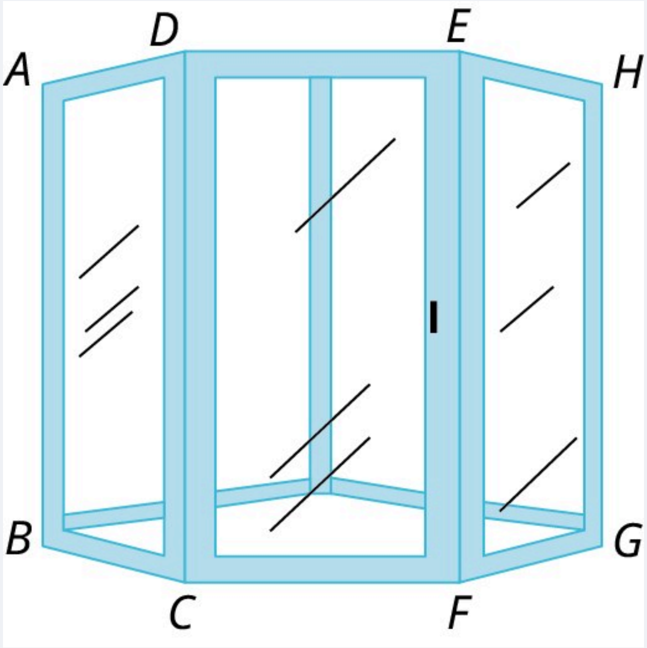
1. Identify the location of points A, B, C, and D.
2. Describe the line from point A to point D.
3. Describe the line from point C containing point B.
4. Does this figure represent a plane?

Solution

1. Point A is located at $(4, 3)$; point B is located at $(-4, 1)$; point C is located at $(-2, -3)$; point D is located at $(1, 1)$.
2. The line from point A to point D is a line segment $\overline{AD}$.
3. The line from point C containing point B is a ray $\overrightarrow{CB}$ starting at point C in the direction of B.
4. Yes, this figure represents a plane because it contains at least three points, points A and D form a line segment, and neither point B nor point C is on that line segment.

Example 6 **Intersecting Planes**

Name two pairs of intersecting planes on the shower enclosure illustration .



Solution

The plane *ABCD* intersects plane *CDEF*, and plane *CDEF* intersects plane *EFGH*.

PEOPLE IN MATHEMATICS

Plato

Part of a remarkable chain of Greek mathematicians, Plato (427–347 BC) is known as the teacher. He was responsible for shaping the development of Western thought perhaps more powerfully than anyone of his time. One of his greatest achievements was the founding of the Academy in Athens where he emphasized the study of geometry. Geometry was considered by the Greeks to be the "ultimate human endeavor." Above the doorway to the Academy, an inscription read, "Let no one ignorant of geometry enter here."

The curriculum of the Academy was a 15-year program. The students studied the exact sciences for the first 10 years. Plato believed that this was the necessary foundation for preparing students' minds to study relationships that require abstract thinking. The next 5 years were devoted to the study of the "dialectic." The dialectic is the art of question and answer. In Plato's view, this skill was critical to the investigation and demonstration of innate mathematical truths. By training young students how to prove propositions and test hypotheses, he created a culture in which the systematic process was guaranteed. The Academy was essentially the world's first university and held the reputation as the ultimate center of learning for more than 900 years.

Key Terms

- line segment
- plane
- union
- intersection
- parallel
- perpendicular

Key Concepts

- Modern-day geometry began in approximately 300 BCE with Euclid's *Elements*, where he defined the principles associated with the line, the point, and the plane.
- Parallel lines have the same slope. Perpendicular lines have slopes that are negative reciprocals of each other.
- The union of two sets, A and B, contains all points that are in both sets and is symbolized as $A \cup B$.
- The intersection of two sets A and B includes only the points common to both sets and is symbolized as $A \cap B$.

10.2 Angles

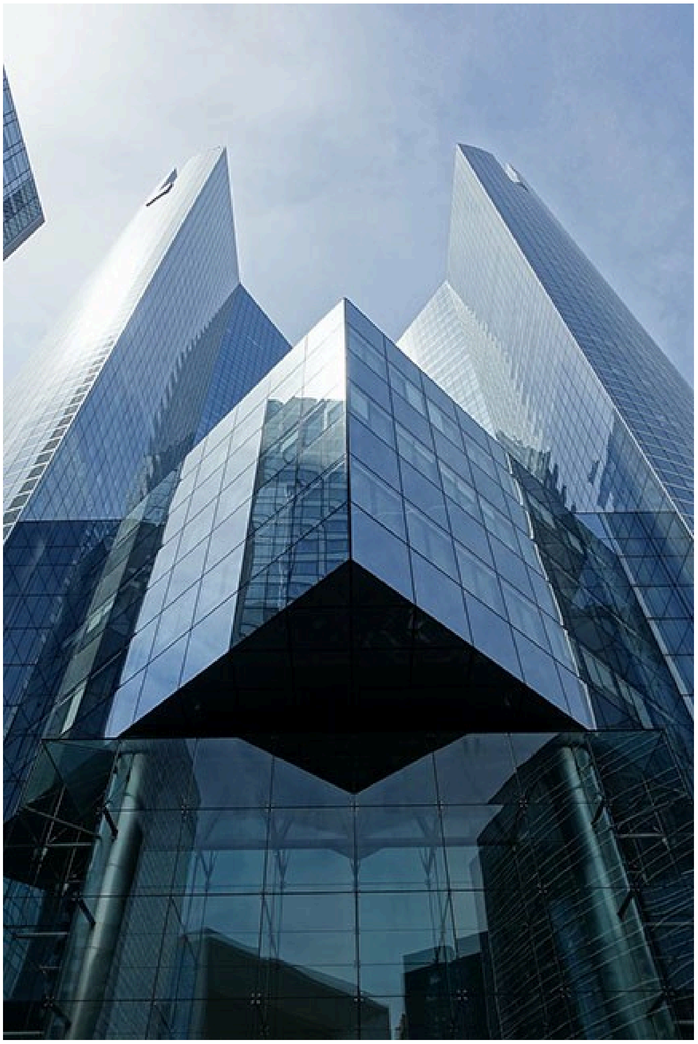


Figure 21: This modern architectural design emphasizes sharp reflective angles as part of the aesthetic through the use of glass walls.

Learning Objectives

After completing this section, you should be able to:

1. Identify and express angles using proper notation.
2. Classify angles by their measurement.
3. Solve application problems involving angles.
4. Compute angles formed by transversals to parallel lines.
5. Solve application problems involving angles formed by parallel lines.

Unusual perspectives on architecture can reveal some extremely creative images. For example, aerial views of cities reveal some exciting and unexpected angles. Add reflections on glass or steel, lighting, and impressive textures, and the structure is a work of art. Understanding angles is critical to many fields, including engineering, architecture, landscaping, space planning, and so on. This is the topic of this section.

We begin our study of angles with a description of how angles are formed and how they are classified. An angle is the joining of two rays, which sweep out as the sides of the angle, with a common endpoint. The common endpoint is called the **vertex**. We will often need to refer to more than one vertex, so you will want to know the plural of vertex, which is vertices.

In , let the ray $\overrightarrow{AB}$ stay put. Rotate the second ray $\overrightarrow{AC}$ in a counterclockwise direction to the size of the angle you want. The angle is formed by the amount of rotation of the second ray. When the ray $\overrightarrow{AC}$ continues to rotate in a counterclockwise direction back to its original position coinciding with ray $\overrightarrow{AB}$, the ray will have swept out 360°. We call the rays the "sides" of the angle.

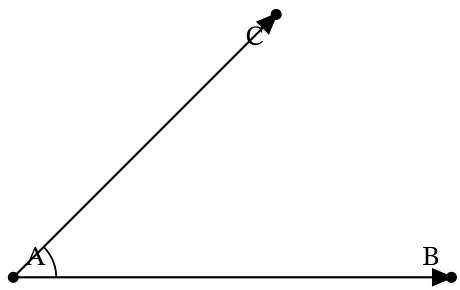


Figure 22: Vertex and Sides of an Angle

Classifying Angles

Angles are measured in **radians** or **degrees**. For example, an angle that measures π radians, or 3.14159 radians, is equal to the angle measuring 180°. An angle measuring $\frac{\pi}{2}$ radians, or 1.570796 radians, measures 90°. To translate degrees to radians, we multiply the angle measure in degrees by $\frac{\pi}{180}$. For example, to write 45° in radians, we have

$$45°\left(\frac{\pi}{180}\right) = \frac{\pi}{4} = 0.785398 \text{ radians.}$$

To translate radians to degrees, we multiply by $\frac{180}{\pi}$. For example, to write 2π radians in degrees, we have

$$2\pi\left(\frac{180}{\pi}\right) = 360°.$$

Another example of translating radians to degrees and degrees to radians is $\frac{2\pi}{3}$. To write in degrees, we have $\frac{2\pi}{3}\left(\frac{180}{\pi}\right) = 120°$. To write 30° in radians, we have $30°\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$. However, we will use degrees throughout this chapter.

FORMULA

To translate an angle measured in degrees to radians, multiply by $\frac{\pi}{180}$.

To translate an angle measured in radians to degrees, multiply by $\frac{180}{\pi}$.

Several angles are referred to so often that they have been given special names. A **straight angle** measures 180° a **right angle** measures 90°; an **acute angle** is any angle whose measure is less than 90°; and an **obtuse angle** is any angle whose measure is between 90° and 180°.

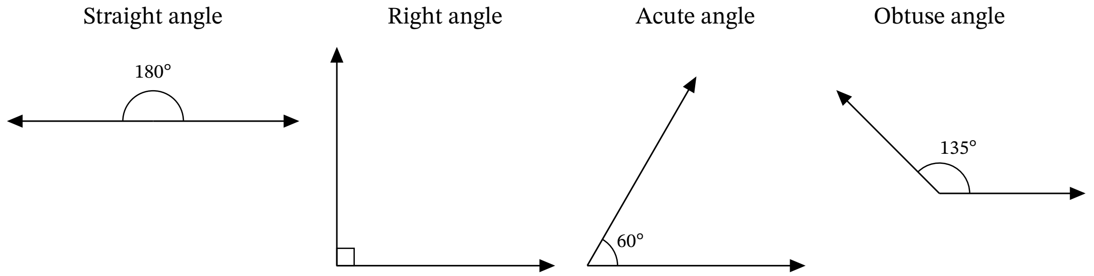


Figure 23: Classifying and Naming Angles

An easy way to measure angles is with a protractor . A protractor is a very handy little tool, usually made of transparent plastic, like the one shown here.

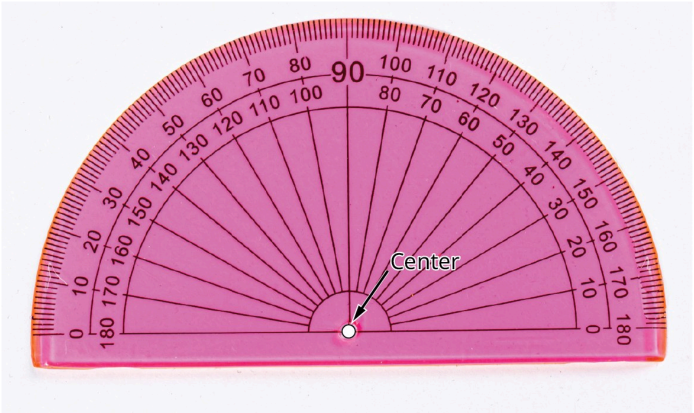


Figure 24: Protractor

With a protractor, you line up the straight bottom with the horizontal straight line of the angle. Be sure to have the center hole lined up with the vertex of the angle. Then, look for the mark on the protractor where the second ray lines up. As you can see from the image, the degrees are marked off. Where the second ray lines up is the measurement of the angle.

CHECKPOINT

Make sure you correctly match the center mark of the protractor with the vertex of the angle to be measured. Otherwise, you will not get the correct measurement. Also, keep the protractor in a vertical position.

Notation

Naming angles can be done in couple of ways. We can name the angle by three points, one point on each of the sides and the vertex point in the middle, or we can name it by the vertex point alone. Also, we can use the symbols $\angle$ or $\measuredangle$ before the points. When we are referring to the measure of the angle, we use the symbol $m\measuredangle$.

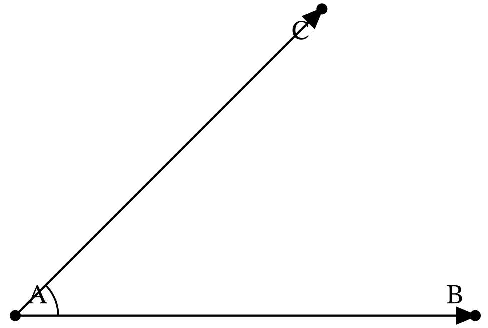
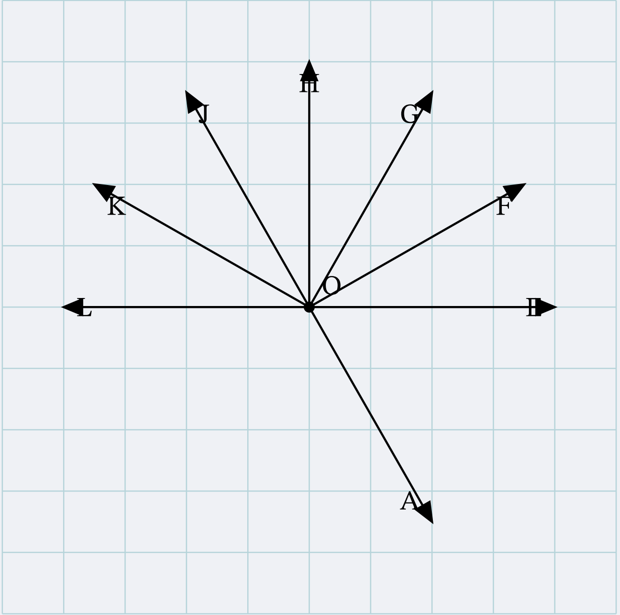


Figure 25: Naming an Angle

We can name this angle $\measuredangle BAC$, or $\measuredangle CAB$, or $\measuredangle A$.

Example 1 **Classifying Angles**

Determine which angles are acute, right, obtuse, or straight on the graph . You may want to use a protractor for this one.



Solution

Acute angles measure less than 90°.	Obtuse angles measure between 90° and 180°.	Right angles measure 90°.	Straight angles measure 180°.
$\angle EOF$ $\angle EOG$ $\angle FOG$ $\angle GOH$ $\angle FOH$ $\angle HOJ$ $\angle HOK$ $\angle JOK$ $\angle KOL$ $\angle JOL$	$\angle EOJ$ $\angle EOK$ $\angle FOJ$ $\angle FOK$ $\angle FOL$ $\angle GOK$ $\angle GOL$	$\angle EOH$ $\angle HOL$ $\angle GOJ$	$\angle EOL$

Most angles can be classified visually or by description. However, if you are unsure, use a protractor.

Adjacent Angles

Two angles with the same starting point or vertex and one common side are called adjacent angles. In , angle $\angle DBC$ is adjacent to $\angle CBA$. Notice that the way we designate an angle is with a point on each of its two sides and the vertex in the middle.

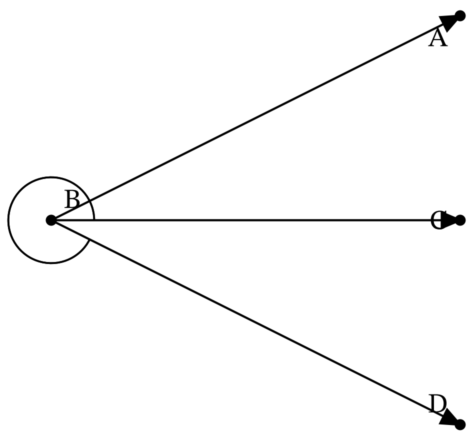


Figure 27: Adjacent Angles

Supplementary Angles

Two angles are **supplementary** if the sum of their measures equals 180°. In , we are given that $m\measuredangle FBE = 35°$, so what is $m\measuredangle ABE$? These are supplementary angles. Therefore, because $m\measuredangle ABF = 180°$, and as $180° - 35° = 145°$, we have $m\measuredangle ABE = 145°$.

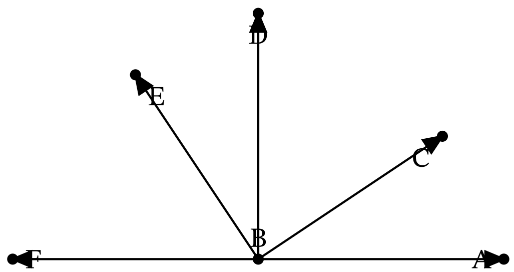
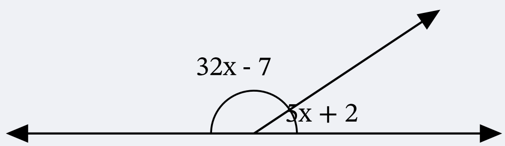


Figure 28: Supplementary Angles

Example 2 **Solving for Angle Measurements and Supplementary Angles**

Solve for the angle measurements in .



Solution

Step 1: These are supplementary angles. We can see this because the two angles are part of a horizontal line, and a horizontal line represents 180°. Therefore, the sum of the two angles equals 180°.

Step 2:

$$\begin{aligned} (32x - 7) + (5x + 2) &= 180 \\ 37x - 5 &= 180 \\ 37x &= 185 \\ x &= \quad 5 \end{aligned}$$

Step 3: Find the measure of each angle:

$$\begin{aligned} 32x - 7 &= 32(5) - 7 \\ &= \qquad 153^\circ \\ 5x + 2 &= \ 5(5) + 2 \\ &= \qquad 27^\circ \end{aligned}$$

Step 4: We check: $153^\circ + 27^\circ = 180^\circ$.

Complementary Angles

Two angles are **complementary** if the sum of their measures equals 90°. In , we have $m\measuredangle ABC = 30^\circ$, and $m\measuredangle ABD = 90^\circ$. What is the $m\measuredangle CBD$? These are complementary angles. Therefore, because $90^\circ - 30^\circ = 60^\circ$, the $\measuredangle CBD = 60^\circ$.

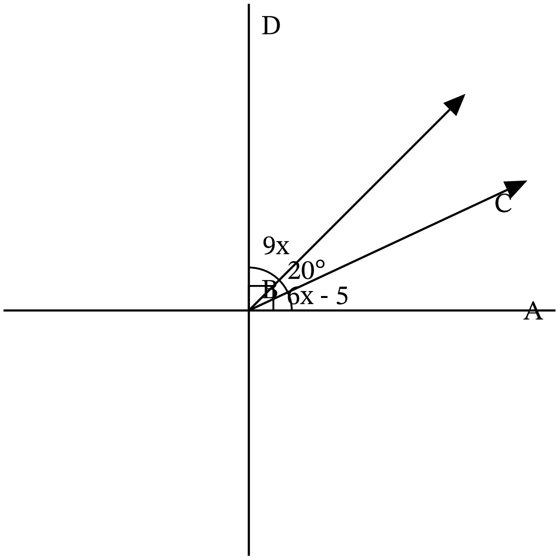
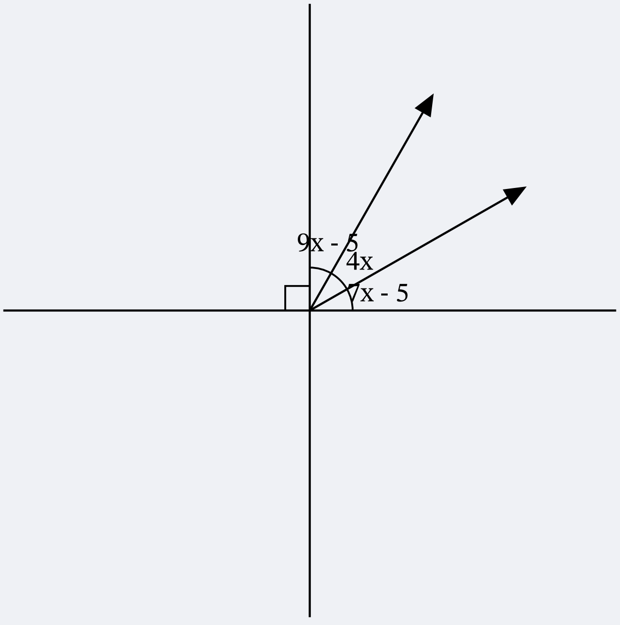


Figure 30: Complementary Angles

Example 3 **Solving for Angle Measurements and Complementary Angles**

Solve for the angle measurements in .



Solution

We have that

$$\begin{aligned}(9x-5)+4x+(7x-5) &= 90\\ 20x &= 100\\ x &= 5\end{aligned}$$

Then, $m\measuredangle(9x-5)=40°$, $m\measuredangle(4x)=20°$, and $m\measuredangle(7x-5)=30°$.

Vertical Angles

When two lines intersect, the opposite angles are called vertical angles, and vertical angles have equal measure. For example, shows two straight lines intersecting each other. One set of opposite angles shows angle markers; those angles have the same measure. The other two opposite angles have the same measure as well.

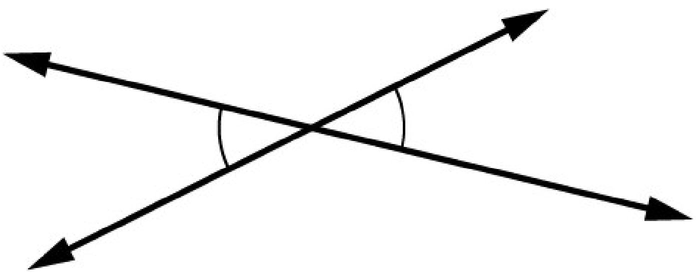
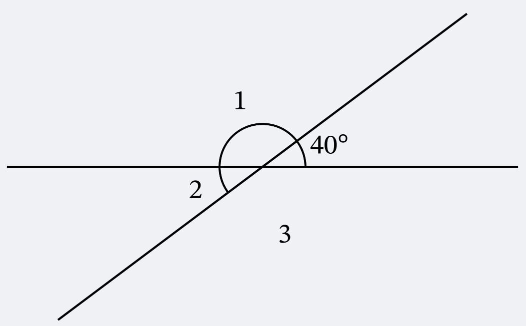


Figure 32: Vertical Angles

Example 4 **Calculating Vertical Angles**

In , one angle measures 40°. Find the measures of the remaining angles.



Solution

The 40-degree angle and $\angle 2$ are vertical angles. Therefore, $m\measuredangle 2=40°$.

Notice that $\measuredangle 2$ and $\measuredangle 1$ are supplementary angles, meaning that the sum of $m\measuredangle 2$ and $m\measuredangle 1$ equals 180°. Therefore, $m\measuredangle 1=180°-40°=140°$.

Since $\measuredangle 1$ and $\measuredangle 3$ are vertical angles, then $m\measuredangle 3$ equals 140°.

Transversals

When two parallel lines are crossed by a straight line or **transversal**, eight angles are formed, including alternate interior angles, alternate exterior angles, corresponding angles, vertical angles, and supplementary angles. Angles 1, 2, 7, and 8 are called exterior angles, and angles 3, 4, 5, and 6 are called interior angles.

Parallel lines intersected by a transversal

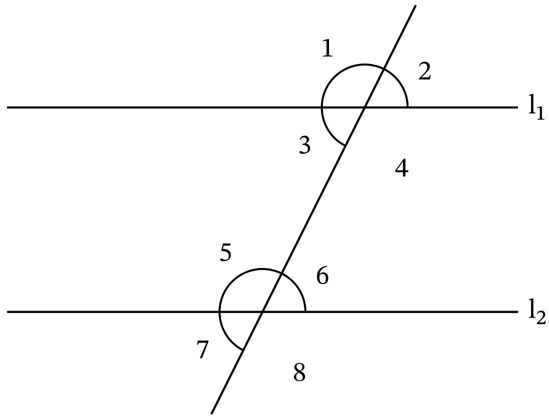


Figure 34: Transversal

Alternate Interior Angles

Alternate interior angles are the interior angles on opposite sides of the transversal. These two angles have the same measure. For example, ∡3 and ∡6 are alternate interior angles and have equal measure; ∡4 and ∡5 are alternate interior angles and have equal measure as well.

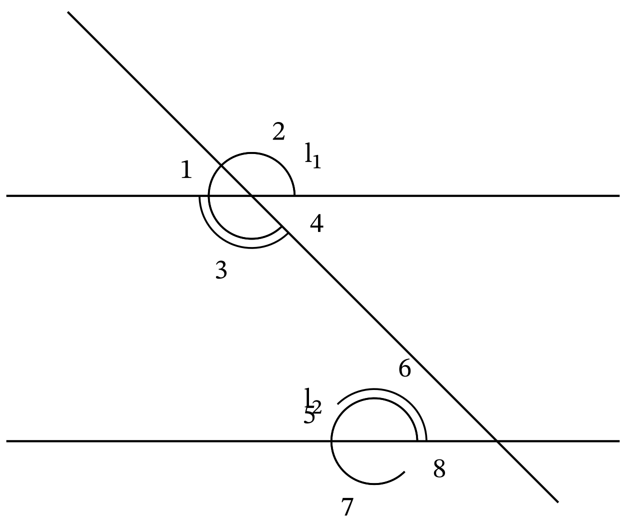


Figure 35: Alternate Interior Angles

Alternate Exterior Angles

Alternate exterior angles are exterior angles on opposite sides of the transversal and have the same measure. For example, in , ∡2 and ∡7 are alternate exterior angles and have equal measures; ∡1 and ∡8 are alternate exterior angles and have equal measures as well.

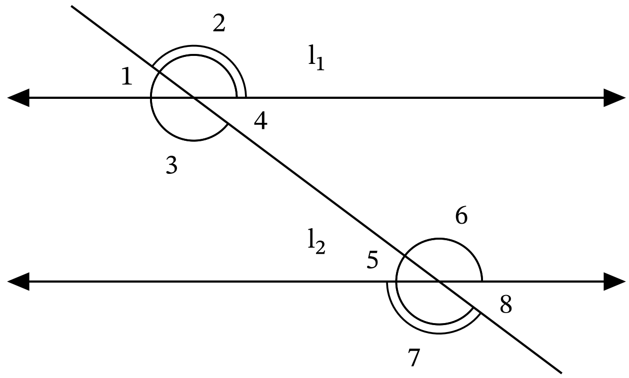


Figure 36: Alternate Exterior Angles

Corresponding Angles

Corresponding angles refer to one exterior angle and one interior angle on the same side as the transversal, which have equal measures. In , ∡1 and ∡5 are corresponding angles and have equal measures; ∡3 and ∡7 are corresponding angles and have equal measures; ∡2 and ∡6 are corresponding angles and have equal measures; ∡4 and ∡8 are corresponding angles and have equal measures as well.

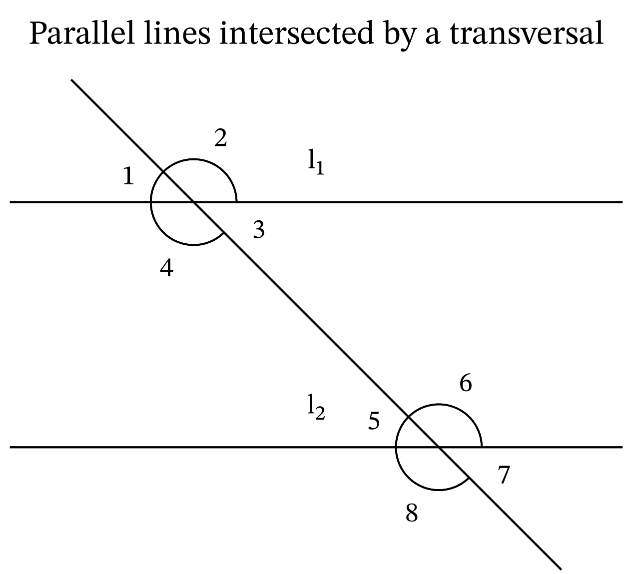
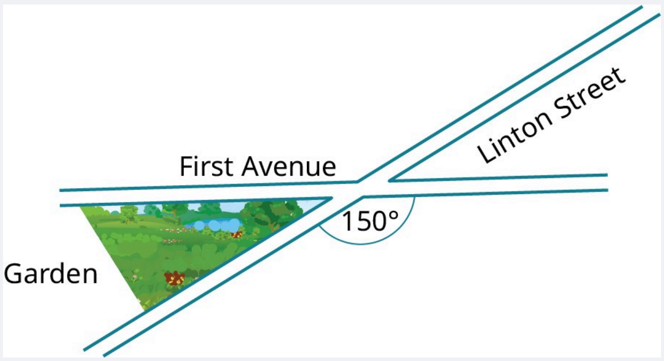


Figure 37: Corresponding Angles

Example 5 **Evaluating Space**

You live on the corner of First Avenue and Linton Street. You want to plant a garden in the far corner of your property and fence off the area. However, the corner of your property does not form the traditional right angle. You learned from the city that the streets cross at an angle equal to 150°. What is the measure of the angle that will border your garden?

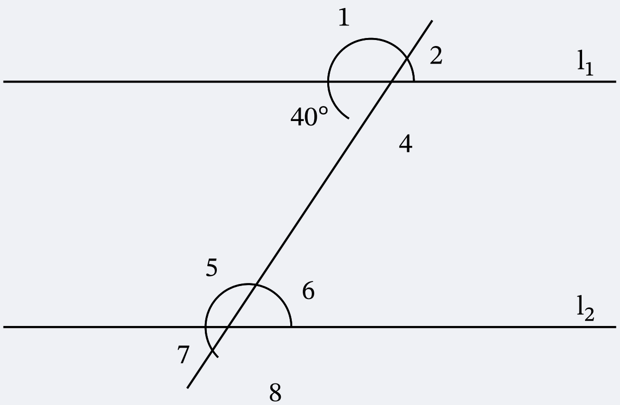


Solution

As the angle between Linton Street and First Avenue is 150°, the supplementary angle is 30°. Therefore, the garden will form a 30° angle at the corner of your property.

Example 6 **Determining Angles Formed by a Transversal**

In given that angle 3 measures 40°, find the measures of the remaining angles and give a reason for your solution.



Solution

$m\measuredangle 2 = m\measuredangle 3 = 40°$ by vertical angles.

$\measuredangle 3 = m\measuredangle 7$ by corresponding angles.

$m\measuredangle 7 = m\measuredangle 6 = 40°$ by vertical angles.

$m\measuredangle 1 = 180 - 40 = 140°$ by supplementary angles.

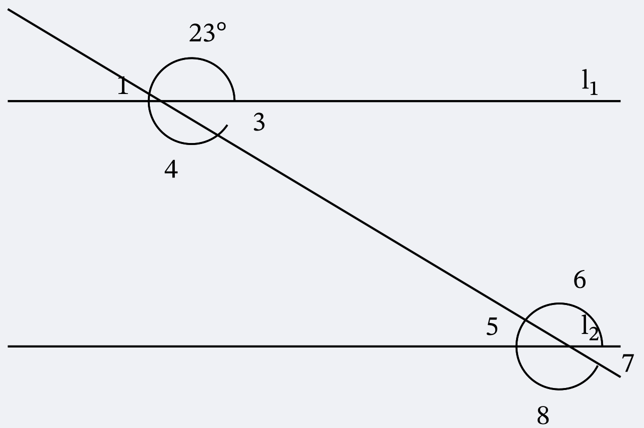
$m\measuredangle 4 = m\measuredangle 1 = 140°$ by vertical angles.

$m\measuredangle 8 = m\measuredangle 1 = 140°$ by alternate exterior angles.

$m\measuredangle 5 = m\measuredangle 8 = 140°$ by vertical angles.

Example 7 **Measuring Angles Formed by a Transversal**

In given that angle 2 measures 23°, find the measure of the remaining angles and state the reason for your solution.



Solution

$m\measuredangle 2 = m\measuredangle 3 = 23°$ by vertical angles, because $\measuredangle 2$ and $\measuredangle 3$ are the opposite angles formed by two intersecting lines.

$m\measuredangle 1 = 157°$ by supplementary angles to $m\measuredangle 2$ or $m\measuredangle 3$. We see that $\measuredangle 1$ and $\measuredangle 2$ form a straight angle as does $\measuredangle 1$ and $\measuredangle 3$. A straight angle measures 180°, so $180° - 23° = 157°$.

$m\measuredangle 4 = m\measuredangle 1 = 157°$ by vertical angles, because $\measuredangle 4$ and $\measuredangle 1$ are the two opposite angles formed by two intersecting lines.

$m\measuredangle 5 = m\measuredangle 1 = 157°$ by corresponding angles because they are the same angle formed by the transversal crossing two parallel lines, one exterior and one interior.

$m\measuredangle 8 = m\measuredangle 5 = 157°$ by vertical angles because $\measuredangle 8$ and $\measuredangle 5$ are the two opposite angles formed by two intersecting lines.

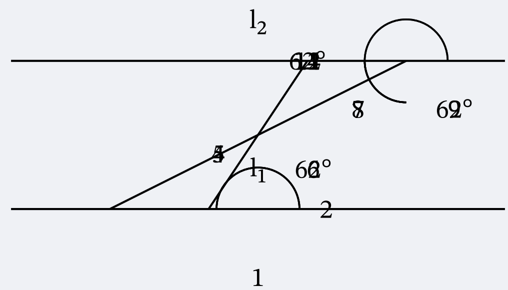
$m\measuredangle 7 = m\measuredangle 2 = 23°$ by alternate exterior angles because, like vertical angles, these angles are the opposite angles formed by the transversal intersecting two parallel lines.

$m\measuredangle 6 = m\measuredangle 7 = 23°$ by vertical angles because these are the opposite angles formed by two intersecting lines.

Example 8 **Finding Missing Angles**

Find the measures of the angles 1, 2, 4, 11, 12, and 14 in and the reason for your answer given that l_1 and l_2 are parallel.

Two parallel lines intersected by two transversals



Solution

$m\measuredangle 12 = 118°$, supplementary angles

$m\measuredangle 14 = 118°$, vertical angles

$m\measuredangle 11 = 62°$, vertical angles

$m\measuredangle 4 = 62°$, corresponding angles

$m\measuredangle 1 = 62°$, vertical angles

$m\measuredangle 2 = 56°$, supplementary angles

WHO KNEW?

The Number 360

Did you ever wonder why there are 360° in a circle? Why not 100° or 500°? The number 360 was chosen by Babylonian astronomers before the ancient Greeks as the number to represent how many degrees in one complete rotation around a circle. It is said that they chose 360 for a couple of reasons: It is close to the number of days in a year, and 360 is divisible by 2, 3, 4, 5, 6, 8, 9, 10, ...

Key Terms

- vertex
- right angle
- acute angle
- obtuse angle
- straight angle
- complementary
- supplementary

Key Concepts

- Angles are classified as acute if they measure less than 90°, obtuse if they measure greater than 90° and less than 180°, right if they measure exactly 90°, and straight if they measure exactly 180°.
- If the sum of angles equals 90°, they are complimentary angles. If the sum of angles equals 180°, they are supplementary.
- A transversal crossing two parallel lines form a series of equal angles: alternate interior angles, alternate exterior angles, vertical angles, and corresponding angles

Formula

To translate an angle measured in degrees to radians, multiply by $\frac{\pi}{180}$.
To translate an angle measured in radians to degrees, multiply by $\frac{180}{\pi}$.

10.3 Triangles

Figure 42: The appearance of triangles in buildings is part of modern-day architectural design.

Learning Objectives

After completing this section, you should be able to:

1. Identify triangles by their sides.
2. Identify triangles by their angles.
3. Determine if triangles are congruent.
4. Determine if triangles are similar.
5. Find the missing side of similar triangles.

How were the ancient Greeks able to calculate the radius of Earth? How did soldiers gauge their target? How was it possible centuries ago to estimate the height of a sail at sea? Triangles have always played a significant role in how we find heights of objects too high to measure or distances between objects too far away to calculate. In particular, the concept of similar triangles has countless applications in the real world, and we shall explore some of those applications in this section.

Technology has given us instruments that allow us to find measurements of distant objects with little effort. However, it is all based on the properties of triangles discovered centuries ago. In this section, we will explore the various types of triangles and their special properties, as well as how to measure interior and exterior angles. We will also explore congruence theorems and similarity.

Identifying Triangles

Joining any three noncollinear points with line segments produces a triangle. For example, given points A, B, and C, connected by the line segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, we have a triangle, as shown.

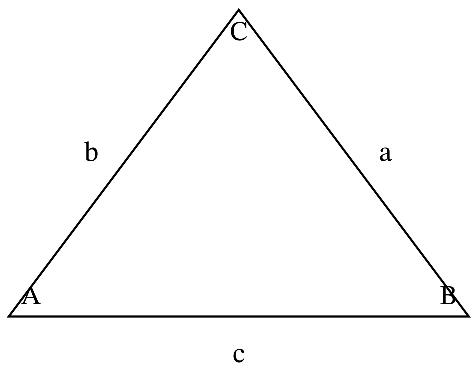


Figure 43: Triangle

Triangles are classified by their angles and their sides. All angles in an acute triangle measure $< 90°$. One of the angles in a right triangle measures 90°, symbolized by ◽. One angle in an obtuse triangle measures between 90° and 180°. Sides that have equal length are indicated by the same hash marks. illustrates the shapes of the basic triangles, their names, and their properties.

A few other facts to remember as we move forward:

- The points where the line segments meet are called the **vertices** (plural for **vertex**).
- We often refer to sides of a triangle by the angle they are opposite. In other words, side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C.

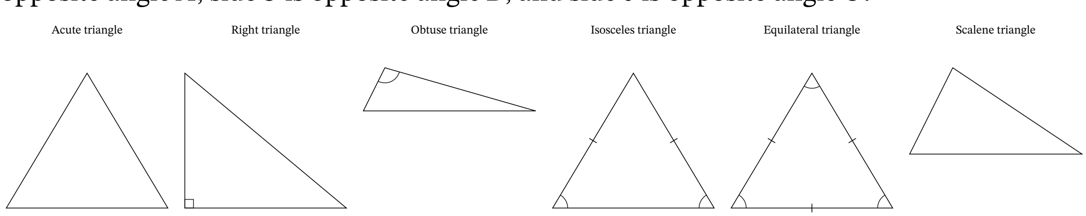


Figure 44: Types of Triangles

We want to add a special note about right triangles here, as they are referred to more than any other triangle. The side opposite the right angle is its longest side and is called the **hypotenuse**, and the sides adjacent to the right angle are called the legs.

One of the most important properties of triangles is that the sum of the interior angles equals 180°. Euclid discovered and proved this property using parallel lines. The completed sketch is shown.

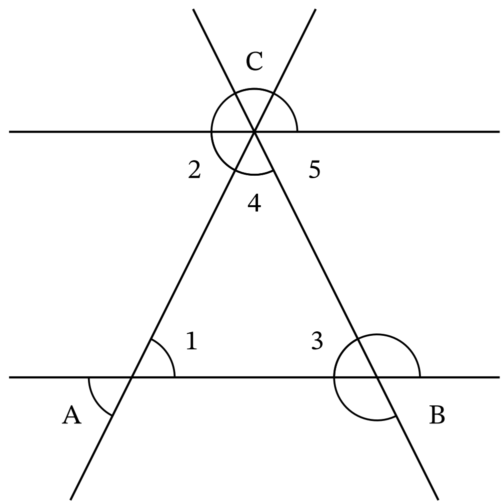


Figure 45: Sum of Interior Angles

This is how the proof goes:

Step 1: Start with a straight line $\overleftrightarrow{AB}$ and a point C not on the line.

Step 2: Draw a line through point C parallel to the line $\overleftrightarrow{AB}$.

Step 3: Construct two transversals (a line crossing the parallel lines), one angled to the right and one angled to the left, to intersect the parallel lines.

Step 4: Because of the property that alternate interior angles inside parallel lines are equal, we have that

$$m\measuredangle 2 = m\sphericalangle 1 \quad \text{and} \quad m\measuredangle 3 = m\measuredangle 4.$$

Step 5: Notice that $m\measuredangle 2 + m\measuredangle 5 + m\measuredangle 4 = 180°$ by the straight angle property.

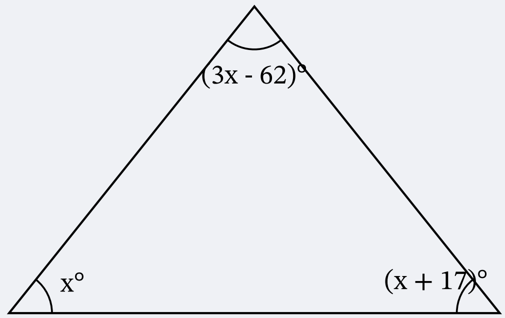
Step 6: Therefore, by substitution, $m\measuredangle 1 = m\measuredangle 2$, and $m\measuredangle 3 = m\measuredangle 4$, we have that

$$m\measuredangle 1 + m\measuredangle 3 + m\sphericalangle 5 = 180° \text{ .}$$

Therefore, the sum of the interior angles of a triangle = 180°.

Example 1 **Finding Measures of Angles Inside a Triangle**

Find the measure of each angle in the triangle shown . We know that the sum of the angles must equal 180°.



Solution

Step 1: As the sum of the interior angles equals 180°, we can use algebra to find the measures:

$$\begin{aligned} x+(x+17)+(3x-62) &= 180 \\ 5x-45 &= 180 \\ 5x &= 225 \\ x &= \frac{225}{5}=45 \end{aligned}$$

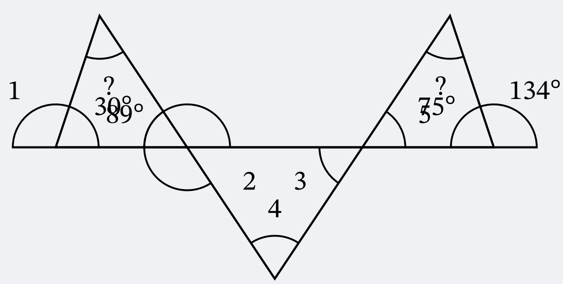
Step 2: Now that we have the value of x, we can substitute 45 into the other two expressions to find the measure of those angles:

$$\begin{aligned} (x+17) &= 45+17=62^\circ \\ (3x-62) &= 3(45)-62=135-62=73^\circ \end{aligned}$$

Step 3: Then, $m\measuredangle x=45^\circ$, $m\measuredangle(x+17)=62^\circ$, and $m\measuredangle(3x-62)=73^\circ$.

Example 2 **Finding Angle Measures**

Find the measure of angles numbered 1–5 in .



Solution

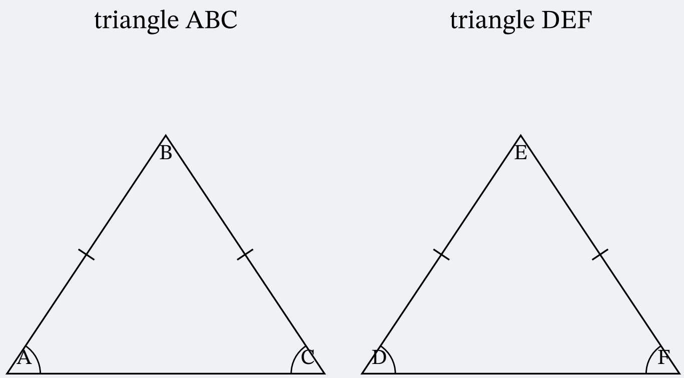
The $m\measuredangle 1=119^\circ$ because it is supplementary with the unknown angle of the adjacent triangle. The unknown angle measures 61°. The $m\measuredangle 2=61^\circ$ because of vertical angles. The $m\measuredangle 5=59^\circ$ because the angle that is supplementary to the 134° measures 46°, and angle 5 is the unknown angle in that triangle. The $m\measuredangle 4=59^\circ$ by vertical angles. Finally, $m\measuredangle 3=60^\circ$, as it is the third angle in the triangle with angles measuring 59° and 61°.

Congruence

If two triangles have equal angles and their sides lengths are equal, the triangles are congruent. In other words, if you can pick up one triangle and place it on top of the other triangle and they coincide, even if you have to rotate one, they are congruent.

Example 3 **Determining If Triangles Are Congruent**

In , is the triangle *ABC* congruent to triangle *DEF*?



Solution

Triangle *ABC* is congruent to triangle *DEF*. Angles *A* and *C* are congruent to angles *D* and *F*, which implies that angle *B* is congruent to angle *E*. Side *AB* is congruent to side *DE*, and side *CB* is congruent to side *FE*, which implies that side *AC* is congruent to side *DF*.

The Congruence Theorems

The following theorems are tools you can use to prove that two triangles are congruent. We use the symbol ≅ to define congruence. For example, $\Delta\, ABC \cong \Delta\, DEF$.

Side-Side-Side (SSS). If three sides of one triangle are equal to the corresponding sides of the second triangle, then the triangles are congruent.

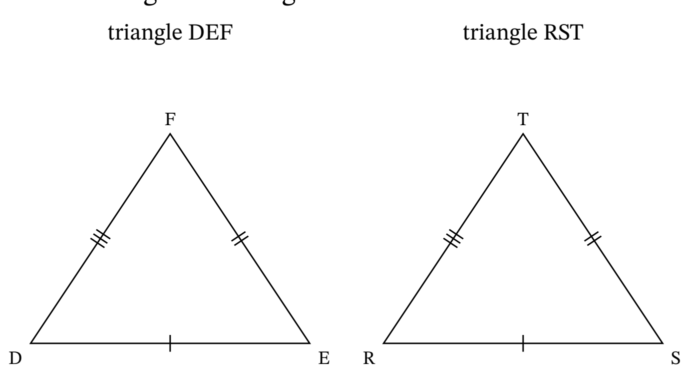


Figure 49: Side-Side-Side (SSS)

We have that $\overline{DF} \cong \overline{RT}$, $\overline{EF} \cong \overline{ST}$, and $\overline{DE} \cong \overline{RS}$, then $\Delta\, DEF \cong \Delta\, RST$.

Side-Angle-Side (SAS). If two sides of a triangle and the angle between them are equal to the corresponding two sides and included angle of the second triangle, then the triangles are congruent. We see that $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, $m\measuredangle B = m\measuredangle B'$, then $\Delta\, ABC \cong \Delta\, A'B'C'$.

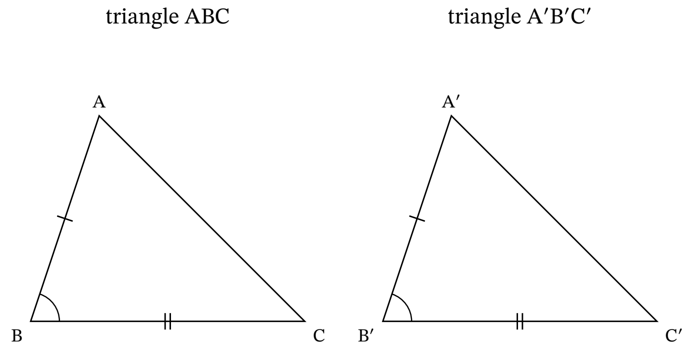


Figure 50: Side-Angle-Side (SAS)

Angle-Side-Angle (ASA). If two angles and the side between them in one triangle are congruent to the two corresponding angles and the side between them in a second triangle, then the two triangles are congruent. Notice that $m\measuredangle$ A $\cong$ $m\measuredangle$ F, and $m\measuredangle$ C $\cong$ $m\measuredangle$ D, $\overline{AC} \cong \overline{DF}$, then $\Delta\, ABC \cong \Delta\, DEF$.

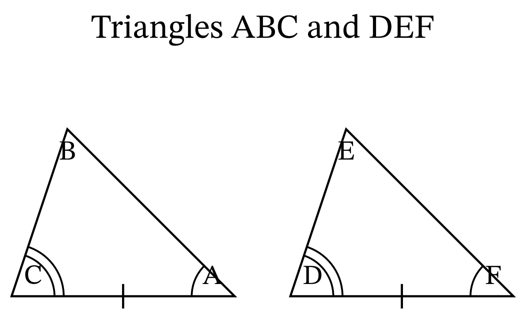


Figure 51: Angle-Side-Angle (ASA)

Angle-Angle-Side (AAS). If two angles and a nonincluded side of one triangle are congruent to two angles and the nonincluded corresponding side of a second triangle, then the triangles are congruent. We see that $m\measuredangle X \cong mX'$, $m\measuredangle Z \cong m\measuredangle Z'$, and $\overline{XY} \cong \overline{X'Y'}$, then $\Delta\, XYZ \cong \Delta\, X'Y'Z'$.

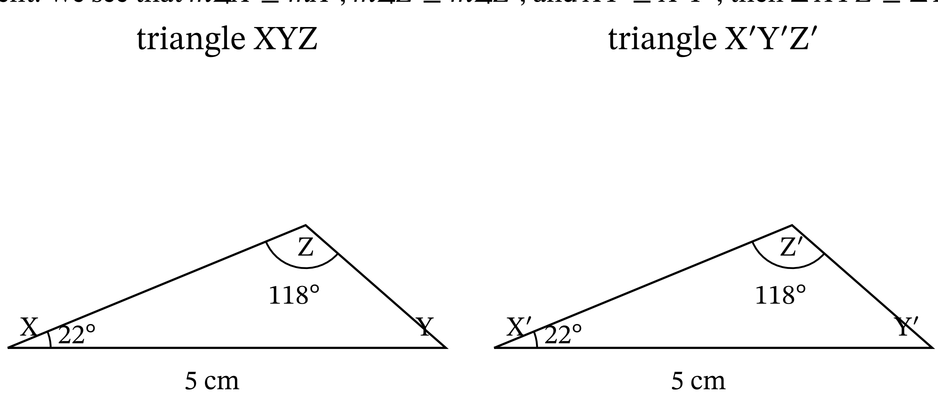
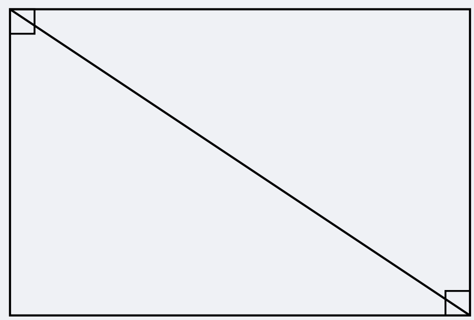


Figure 52: Angle-Angle-Side (AAS)

Example 4 **Identifying Congruence Theorems**

What congruence theorem is illustrated in ?

Rectangle formed by two right triangles

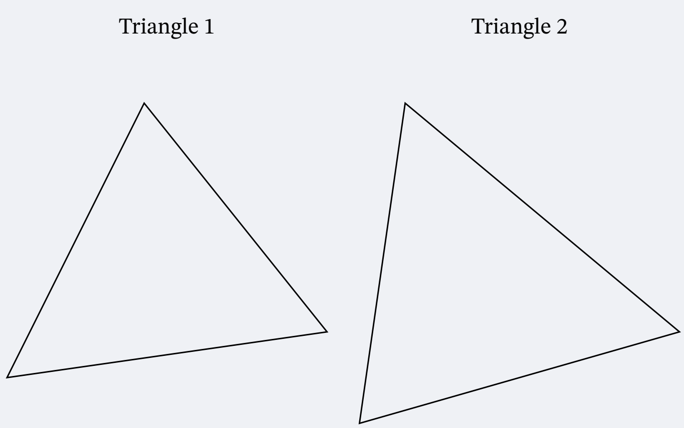


Solution

AAS: Two angles and a non-included side in one triangle are congruent to the corresponding angles and side in the second triangle.

Example 5 **Determining the Congruence Theorem**

What congruence theorem is illustrated in ?



Solution

The SSS theorem.

Similarity

If two triangles have the same angle measurements and are the same shape but differ in size, the two triangles are similar. The lengths of the sides of one triangle will be proportional to the corresponding sides of the second triangle. Note that a single fraction $\frac{a}{b}$ is called a ratio, but two fractions equal to each other is called a proportion, such as $\frac{a}{b} = \frac{c}{d}$.

This rule of similarity applies to all shapes as well as triangles. Another way to view similarity is by applying a **scaling factor**, which is the ratio of corresponding measurements between an object or representation of the object, to an image that produces the second, similar image.

For example, why are the two images in are similar? These two images have the same proportions between elements. Therefore, they are similar.

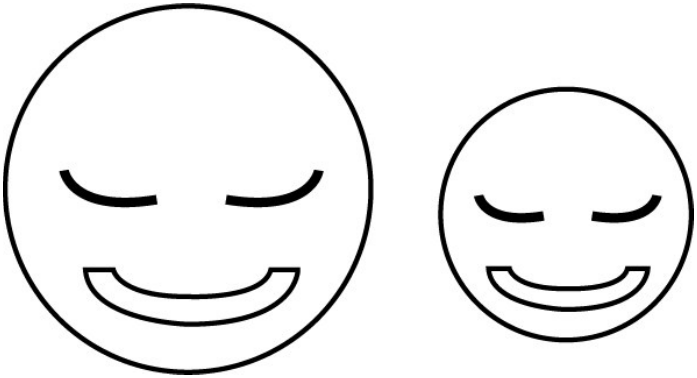
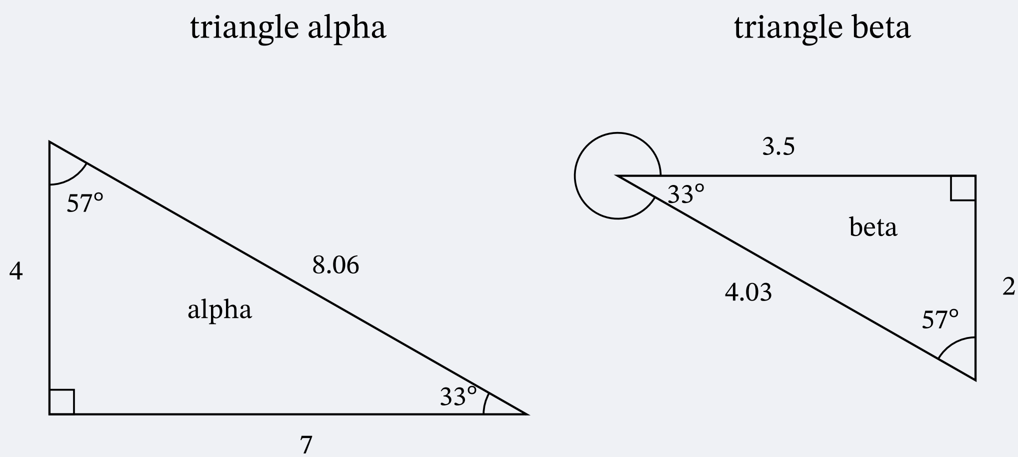


Figure 55: Similarity

Example 6 **Determining If Triangles Are Similar**

Are the two triangles shown in similar?



Solution

Step 1: We will look at the proportions within each triangle. In triangle α (alpha), the side opposite the 57° angle measures 7, and the side opposite the 33° angle measures 4. Then, the measures of the corresponding sides in triangle β (beta) measures 3.5 and 2, respectively. We have

$$\frac{4}{7} = 0.5714 \qquad \frac{2}{3.5} = 0.5714.$$

This is the proportion $\frac{4}{7} = \frac{2}{3.5}$. The scaling factor is 0.5714.

Step 2: Let's try another correspondence. In triangle α, the hypotenuse measures 8.06 and the side opposite the 57° angle measures 7. In triangle β, the hypotenuse measure 4.03 and the side opposite the 57° angle measures 3.5. We have

$$\frac{7}{8.06} = 0.8685 \qquad \frac{3.5}{4.03} = 0.8685.$$

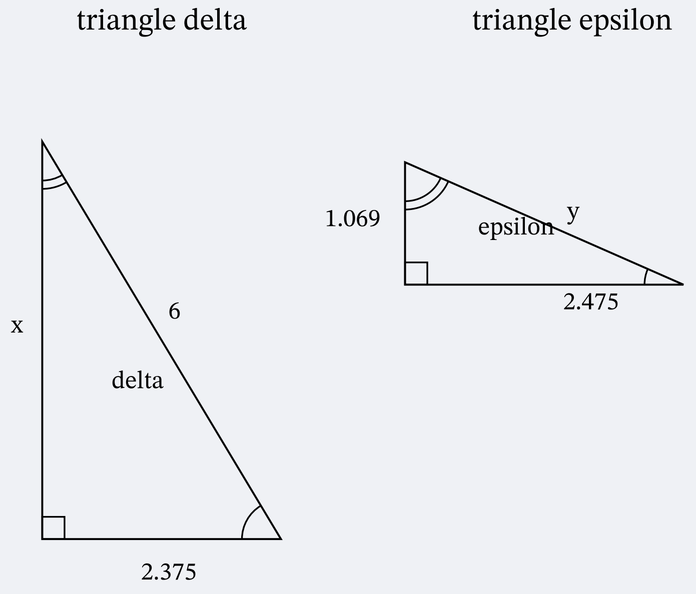
Step 3: Now, let's look at the proportions between triangle α and triangle β. The side measuring 2 in triangle β corresponds to the side measuring 4 in triangle α, the side measuring 3.5 in triangle β corresponds to the side measuring 7 in triangle α, and the hypotenuse in triangle β corresponds to the hypotenuse in triangle α. We have

$$\frac{2}{4} = 0.5 \qquad \frac{3.5}{7} = 0.5 \qquad \frac{4.03}{8.06} = 0.5$$

Thus, the corresponding angles are equal and the proportions between each pair of corresponding sides equals 0.5. In other words, the scaling factor is 0.5. Therefore, the triangles are similar.

Example 7 **Proving Similarity**

In , is triangle δ (delta) similar to triangle ε (epsilon)? Find the lengths of sides x and y as part of your answer.



Solution

We can see that all three angles in triangle δ are equal to the corresponding angles in triangle ε. That is enough to determine similarity. However, we want to find the values of x and y to prove similarity.

Step 1: We have to do is set up the proportions between the corresponding sides. We have the side that measures 2.375 in triangle δ corresponding to the side measuring 1.069 in triangle ε. We have the hypotenuse/side in triangle δ measuring 6 corresponding to the hypotenuse/side labeled y in triangle ε. And, finally, the side labeled x in triangle δ corresponds to the side measuring 2.475 in triangle ε.

Each proportion should be equal. We start with the proportion of the shorter sides. Thus

$$\frac{1.069}{2.375} = 0.45$$

Step 2: We solve for y using the first proportion. Set the two ratios equal to each other, cross-multiply, and solve for y. We have:

$$\begin{aligned} \frac{1.069}{2.375} &= \frac{y}{6} \\ (6)(1.069) &= (2.375)(y) \\ 6.414 &= 2.375y \\ \frac{6.414}{2.375} &= 2.7 = y \end{aligned}$$

So, $y = 2.7$.

Step 3: Checking that length in the proportion factor of 0.45, we have:

$$\frac{y}{6} = \frac{2.7}{6} = 0.45$$

Step 4: Solving for x, we will use the same proportion we used to solve for y. We have:

$$\begin{aligned} \frac{1.069}{2.375} &= \frac{2.475}{x} \\ 1.069(x) &= 2.475(2.375) \\ 1.069(x) &= 5.878 \\ x &= \frac{5.878}{1.069} = 5.5 \\ \frac{2.475}{x} &= \frac{2.475}{5.5} = 0.45 \end{aligned}$$

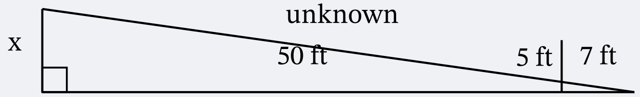
Step 5: We test the proportions. We have the following:

$$\frac{2.7}{6} = \frac{2.475}{5.5} = \frac{1.069}{2.375} = 0.45$$

The proportions are all equal. Therefore, we have proven the property of similarity between triangle δ and triangle ε.

Example 8 **Applying Similar Triangles**

A person who is 5 feet tall is standing 50 feet away from the base of a tree . The tree casts a 57-foot shadow. The person casts a 7-foot shadow. What is the height of the tree?



Solution

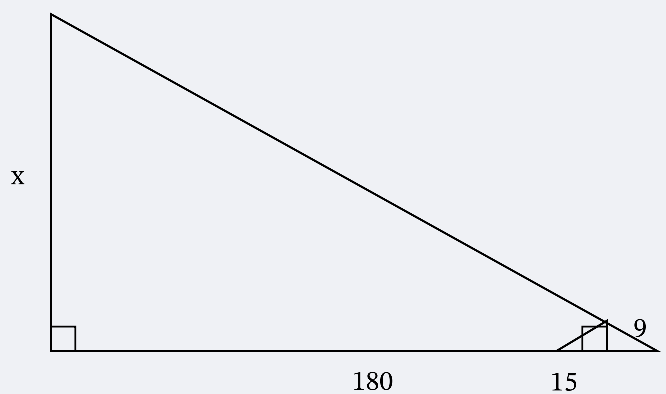
The bigger triangle includes a tree at side x and the smaller triangle includes the person at the side labeled 5 ft. These two triangles are similar because the smaller triangle fits inside the larger triangle at the smallest angle. It would fit inside the larger triangle at either of the other two angles as well. That all angles are equal is one of the criteria for similar triangles, so we can solve using proportions:

$$\begin{aligned} \frac{5}{7} &= \frac{x}{57} \\ 5(57) &= 7x \\ \frac{285}{7} &= x = 40.7 \end{aligned}$$

The tree is 40.7 feet tall.

Example 9 **Finding Missing Lengths**

At a certain time of day, a radio tower casts a shadow 180 feet long . At the same time, a 9-foot truck casts a shadow 15 feet long. What is the height of the tower?



Solution

These are similar triangles and the problem can be solved by using proportions:

$$\begin{aligned} \frac{x}{180} &= \frac{9}{15} \\ 9(180) &= 15x \\ 1620 &= 15x \\ 108 &= x \end{aligned}$$

The height of the tower is 108 ft.

PEOPLE IN MATHEMATICS

Thales of Miletus

Thales of Miletus, sixth century BC, is considered one of the greatest mathematicians and philosophers of all time. Thales is credited with being the first to discover that the two angles at the base of an isosceles triangle are equal, and that the two angles formed by intersecting lines are equal—that is, vertical or opposite angles, are equal. Thales is also known for devising a method for measuring the height of the pyramids by similar right triangles. shows his method. He measured the length of the shadow cast by the pyramid at the precise time when his own shadow ended at the same place.

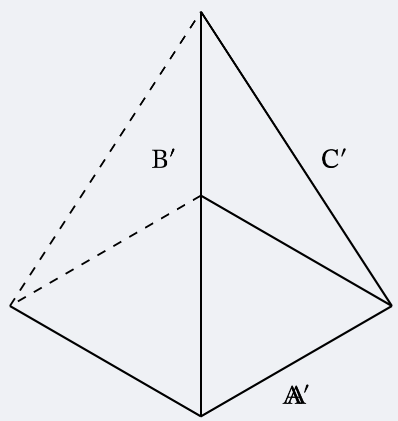


Figure 60: Thales and Similarity

He equated the vertical height of the pyramid with his own height; the horizontal distance from the pyramid to the tip of its shadow with the distance from himself and the tip of his own shadow; and finally, the length of the shadow cast off the top of the pyramid with length of his own shadow cast off the top of his head. Using proportions, as shown in , he essentially discovered the properties of similarity for right triangles. That is, $\triangle ABC$ is similar to $\triangle A'B'C'$. Note that to be similar, all corresponding angles between the two triangles must be equal, and the proportions from one side to another side within each triangle, as well as the proportions of the corresponding sides between the two triangles must be equal.

Thales is also credited with discovering a method of determining the distance of a ship from the shoreline. Here is how he did it, as illustrated in .

Triangles ABC and CDE

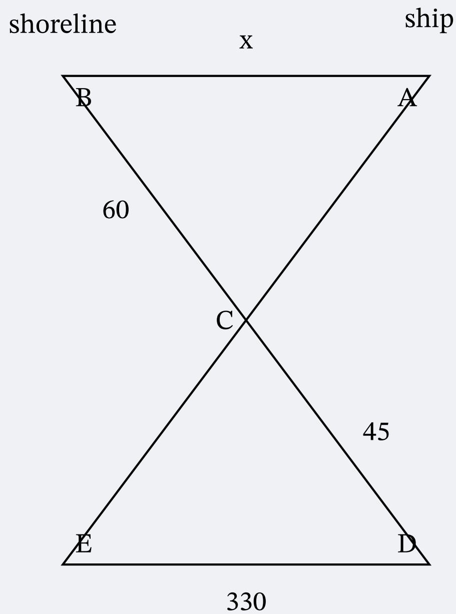


Figure 61: Thales and Similar Triangles

Thales walked along the shoreline pointing a stick at the ship until it formed a 90° angle to the shore. Then he walked along the shot and placed the stick in the ground at point *C*. He continued walking until he reached point *D*. Then, he turned and walked away from the shore at a 90° angle until the stick he placed in the ground at point *C* lined up with the ship, point *E*. This is how he created similar triangles and estimated the distance of the ship to the shore by using proportions.

Key Terms

- acute
- obtuse
- isosceles
- equilateral
- hypotenuse
- congruence
- similarity
- scaling factor

Key Concepts

- The sum of the interior angles of a triangle equals 180°.

- Two triangles are congruent when the corresponding angles have the same measure and the corresponding side lengths are equal.
- The congruence theorems include the following: SAS, two sides and the included angle of one triangle are congruent to the same in a second triangle; ASA, two angles and the included side of one triangle are congruent to the same in a second triangle; SSS, all three side lengths of one triangle are congruent to the same in a second triangle; AAS, two angles and the non-included side of one triangle are congruent to the same in a second triangle.
- Two shapes are similar when the proportions between corresponding angles, sides or features of two shapes are equal, regardless of size.

10.4 Polygons, Perimeter, and Circumference

Figure 62: Geometric patterns are often used in fabrics due to the interest the shapes create.

Learning Objectives

After completing this section, you should be able to:

1. Identify polygons by their sides.
2. Identify polygons by their characteristics.
3. Calculate the perimeter of a polygon.
4. Calculate the sum of the measures of a polygon's interior angles.
5. Calculate the sum of the measures of a polygon's exterior angles.
6. Calculate the circumference of a circle.
7. Solve application problems involving perimeter and circumference.

In our homes, on the road, everywhere we go, polygonal shapes are so common that we cannot count the many uses. Traffic signs, furniture, lighting, clocks, books, computers, phones, and so on, the list is endless. Many applications of polygonal shapes are for practical use, because the shapes chosen are the best for the purpose.

Modern geometric patterns in fabric design have become more popular with time, and they are used for the beauty they lend to the material, the window coverings, the dresses, or the upholstery. This art is not done for any practical reason, but only for the interest these shapes can create, for the pure aesthetics of design.

When designing fabrics, one has to consider the perimeter of the shapes, the triangles, the hexagons, and all polygons used in the pattern, including the circumference of any circular shapes. Additionally, it is the relationship of one object to another and experimenting with different shapes, changing perimeters, or changing angle measurements that we find the best overall design for the intended use of the fabric. In this section, we will explore these properties of polygons, the perimeter, the calculation of interior and exterior angles of polygons, and the circumference of a circle.

Identifying Polygons

A **polygon** is a closed, two-dimensional shape classified by the number of straight-line sides. See for some examples. We show only up to eight-sided polygons, but there are many, many more.

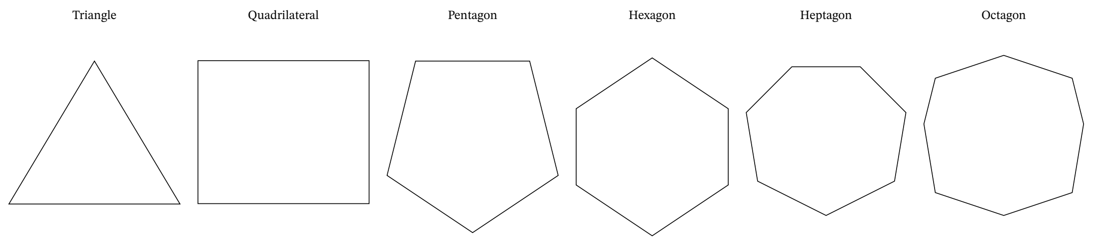


Figure 63: Types of Polygons

If all the sides of a polygon have equal lengths and all the angles are equal, they are called **regular polygons**. However, any shape with sides that are line segments can classify as a polygon. For example, the first two shapes, shown in and , are both pentagons because they each have five sides and five vertices. The third shape is a hexagon because it has six sides and six vertices. We should note here that the hexagon in is a concave hexagon, as opposed to the first two shapes, which are convex pentagons. Technically, what makes a polygon concave is having an interior angle that measures greater than 180°. They are hollowed out, or cave in, so to speak. Convex refers to the opposite effect where the shape is rounded out or pushed out.

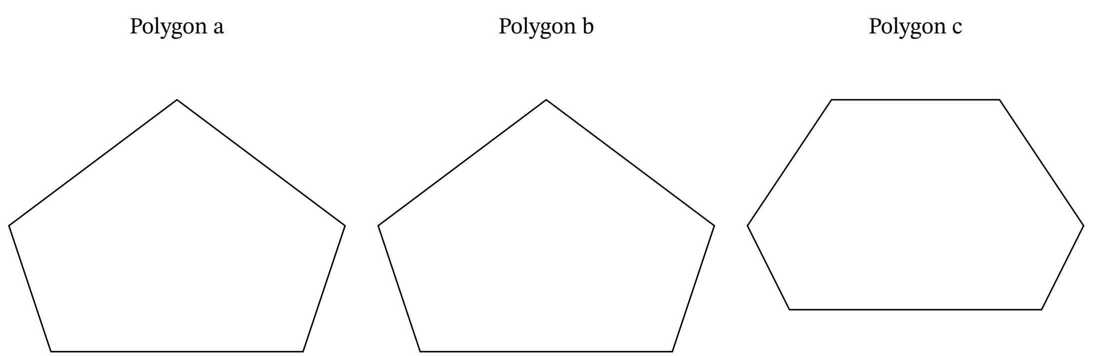


Figure 64: Polygons

While there are variations of all polygons, quadrilaterals contain an additional set of figures classified by angles and whether there are one or more pairs of parallel sides.

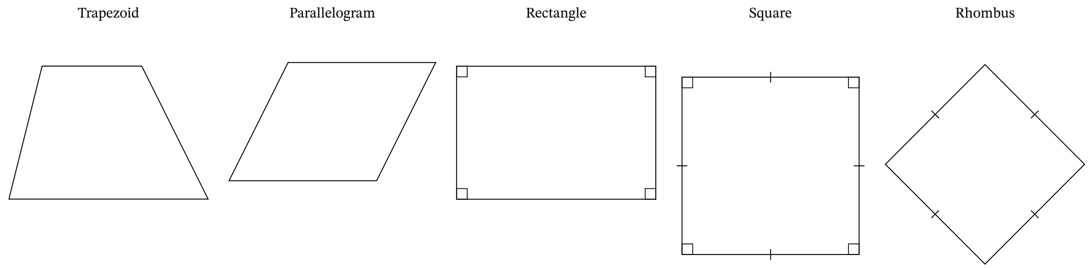


Figure 65: Types of Quadrilaterals

Example 1 **Identifying Polygons**

Identify each polygon.

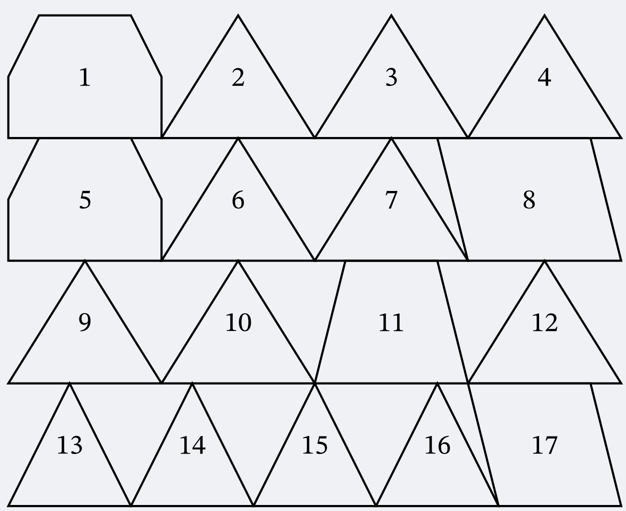
Solution

1. This shape has six sides. Therefore, it is a hexagon.
2. This shape has four sides, so it is a quadrilateral. It has two pairs of parallel sides making it a parallelogram.
3. This shape has eight sides making it an octagon.
4. This is an equilateral triangle, as all three sides are equal.
5. This is a rhombus; all four sides are equal.
6. This is a regular octagon, eight sides of equal length and equal angles.

Example 2 **Determining Multiple Polygons**

What polygons make up ?

Rectangle partitioned into 17 polygons



Solution

Shapes 1 and 5 are hexagons; shapes 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, and 16 are triangles; shapes 8 and 17 are parallelograms; and shape 11 is a trapezoid.

Perimeter

Perimeter refers to the outside measurements of some area or region given in linear units. For example, to find out how much fencing you would need to enclose your backyard, you will need the perimeter. The general definition of **perimeter** is the sum of the lengths of the sides of an enclosed region. For some geometric shapes, such as rectangles and circles, we have formulas. For other shapes, it is a matter of just adding up the side lengths.

A **rectangle** is defined as part of the group known as quadrilaterals, or shapes with four sides. A rectangle has two sets of parallel sides with four angles. To find the perimeter of a rectangle, we use the following formula:

FORMULA

The formula for the perimeter P of a rectangle is $P = 2L + 2W$, twice the length L plus twice the width W.

For example, to find the length of a rectangle that has a perimeter of 24 inches and a width of 4 inches, we use the formula. Thus,

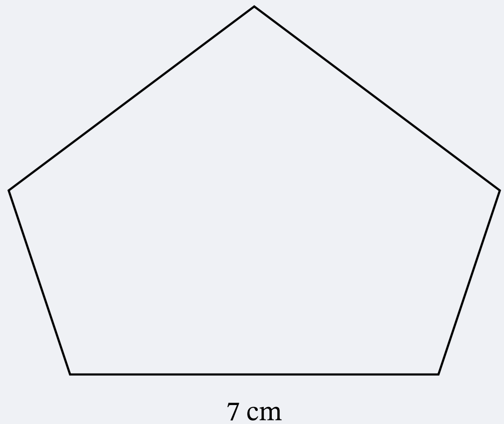
$$\begin{aligned} 24 &= 2l + 2(4) \\ &= \quad 2l + 8 \\ 24 - 8 &= \quad 2l \\ 16 &= \quad 2l \\ 8 &= \quad l \end{aligned}$$

The length is 8 units.

The perimeter of a regular polygon with n sides is given as $P = n \cdot s$. For example, the perimeter of an equilateral triangle, a triangle with three equal sides, and a side length of 7 cm is $P = 3(7) = 21$ cm.

Example 3 **Finding the Perimeter of a Pentagon**

Find the perimeter of a regular pentagon with a side length of 7 cm .

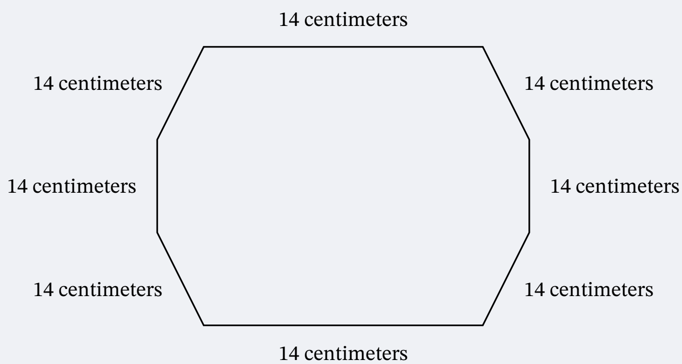


Solution

A regular pentagon has five equal sides. Therefore, the perimeter is equal to $P = 5(7) = 35$ cm.

Example 4 **Finding the Perimeter of an Octagon**

Find the perimeter of a regular octagon with a side length of 14 cm .



Solution

A regular octagon has eight sides of equal length. Therefore, the perimeter of a regular octagon with a side length of 14 cm is $P = 8(14) = 112$ cm.

Sum of Interior and Exterior Angles

To find the sum of the measurements of interior angles of a regular polygon, we have the following formula.

FORMULA

The sum of the interior angles of a polygon with n sides is given by

$$S = (n - 2)180°.$$

For example, if we want to find the sum of the interior angles in a parallelogram, we have

$$\begin{aligned} S &= (4-2)180^\circ \\ &= 2(180) = 360^\circ. \end{aligned}$$

Similarly, to find the sum of the interior angles inside a regular heptagon, we have

$$\begin{aligned} S &= (7-2)180^\circ \\ &= 5(180) \\ &= 900^\circ. \end{aligned}$$

To find the measure of each interior angle of a regular polygon with n sides, we have the following formula.

FORMULA

The measure of each interior angle of a regular polygon with n sides is given by

$$a = \frac{(n-2)180^\circ}{n}.$$

For example, find the measure of an interior angle of a regular heptagon, as shown. We have

$$a = \frac{(7-2)180^\circ}{7} = 128.57^\circ.$$

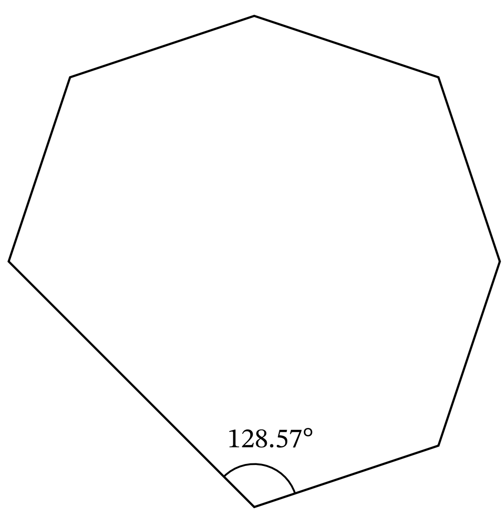


Figure 69: Interior Angles

Example 5 **Calculating the Sum of Interior Angles**

Find the measure of an interior angle in a regular octagon using the formula, and then find the sum of all the interior angles using the sum formula.

Solution

An octagon has eight sides, so $n = 8$.

Step 1: Using the formula $a = \frac{(n-2)180°}{8}$:

$$\begin{aligned} a &= \frac{(8-2)180°}{8} \\ &= \frac{(6)180°}{8} \\ &= 135°. \end{aligned}$$

So, the measure of each interior angle in a regular octagon is 135°.

Step 2: The sum of the angles inside an octagon, so using the formula:

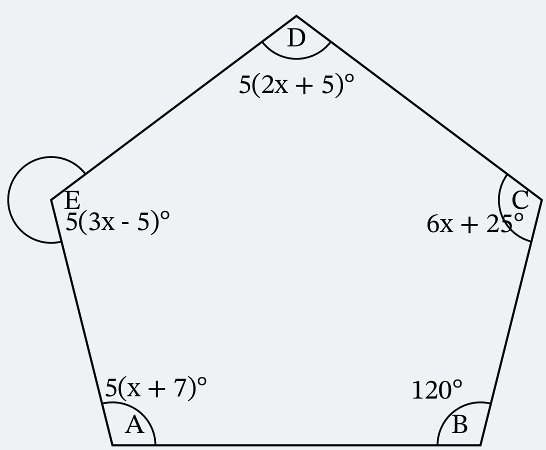
$$\begin{aligned} S &= (n-2)180° \\ &= (8-2)180° \\ &= 6(180) \\ &= 1,080°. \end{aligned}$$

Step 3: We can test this, as we already know the measure of each angle is 135°. Thus, $8(135°) = 1,080°$.

Example 6 **Calculating Interior Angles**

Use algebra to calculate the measure of each interior angle of the five-sided polygon .

pentagon ABCDE



Solution

Step 1: Let us find out what the total of the sum of the interior angles should be. Use the formula for the sum of the angles in a polygon with n sides: $S = (n-2)180°$. So, $S = (5-2)180° = 540°$.

Step 2: We add up all the angles and solve for x:

$$\begin{aligned} 5(x+7)+120+(6x+25)+5(2x+5)+5(3x-5) &= 540 \\ 5x+6x+10x+15x+180 &= 540 \\ 36x &= 360 \\ x &= \ 10 \end{aligned}$$

Step 3: We can then find the measure of each interior angle:

$$m\measuredangle A = 5(10+7) = 85°$$
$$m\measuredangle B = 120°$$
$$m\measuredangle C = 6(10)+25 = 85°$$
$$m\measuredangle D = 5(2*10+5) = 125°$$
$$m\measuredangle E = 5(3*10-5) = 125°$$

An **exterior angle** of a regular polygon is an angle formed by extending a side length beyond the closed figure. The measure of an exterior angle of a regular polygon with n sides is found using the following formula:

FORMULA

To find the measure of an exterior angle of a regular polygon with n sides we use the formula

$$b = \frac{360°}{n}.$$

In , we have a regular hexagon $ABCDEF$. By extending the lines of each side, an angle is formed on the exterior of the hexagon at each vertex. The measure of each exterior angle is found using the formula, $b = \frac{360°}{6} = 60°$.

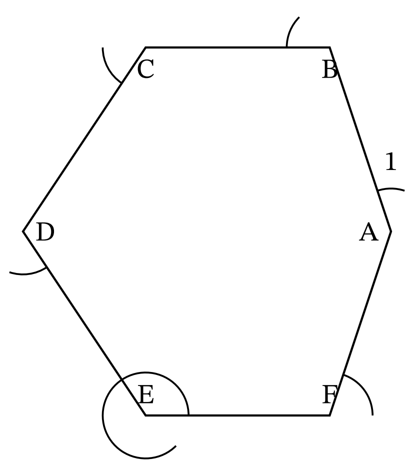
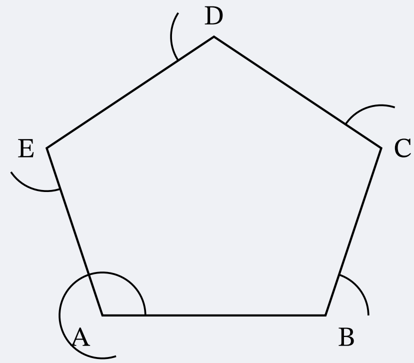


Figure 71: Exterior Angles

Now, an important point is that the **sum** of the exterior angles of a regular polygon with n sides equals 360°. This implies that when we multiply the measure of one exterior angle by the number of sides of the regular polygon, we should get 360°. For the example in , we multiply the measure of each exterior angle, 60°, by the number of sides, six. Thus, the sum of the exterior angles is $6(60°) = 360°$.

Example 7 **Calculating the Sum of Exterior Angles**

Find the sum of the measure of the exterior angles of the pentagon .



Solution

Each individual angle measures $\frac{360}{5} = 72°$. Then, the sum of the exterior angles is $5(72°) = 360°$.

Circles and Circumference

The perimeter of a circle is called the **circumference**. To find the circumference, we use the formula $C = \pi d$, where d is the diameter, the distance across the center, or $C = 2\pi r$, where r is the radius.

FORMULA

The circumference of a circle is found using the formula $C = \pi d$, where d is the diameter of the circle, or $C = 2\pi r$, where r is the radius.

The radius is ½ of the diameter of a circle. The symbol $\pi = 3.141592654\ldots$ is the ratio of the circumference to the diameter. Because this ratio is constant, our formula is accurate for any size circle.

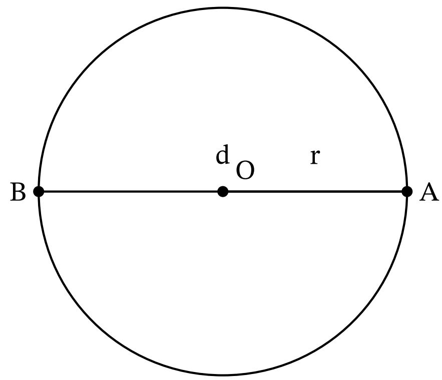


Figure 73: Circle Diameter and Radius

Let the radius be equal to 3.5 inches. Then, the circumference is

$$\begin{aligned} C = & \; 2\pi(3.5) \\ = & \; 21.99 \text{ in} . \end{aligned}$$

Example 8 **Finding Circumference with Diameter**

Find the circumference of a circle with diameter 10 cm.

Solution

If the diameter is 10 cm, the circumference is $C = 10\pi = 31.42$ cm .

Example 9 **Finding Circumference with Radius**

Find the radius of a circle with a circumference of 12 in.

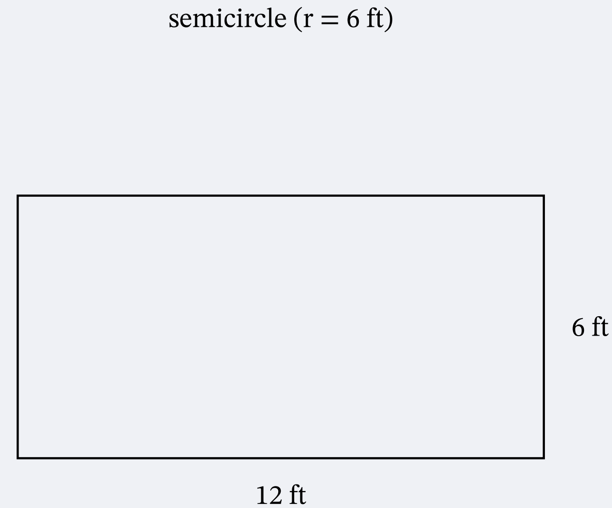
Solution

If the circumference is 12 in, then the radius is

$$\begin{aligned} 12 = & \; 2\pi r \\ \frac{12}{2\pi} = & \; r = 1.91 \text{ in} . \end{aligned}$$

Example 10 **Calculating Circumference for the Real World**

You decide to make a trim for the window in . How many feet of trim do you need to buy?



Solution

The trim will cover the 6 feet along the bottom and the two 12-ft sides plus the half circle on top. The circumference of a semicircle is ½ the circumference of a circle. The diameter of the semicircle is 6 ft. Then, the circumference of the semicircle would be $\frac{1}{2}\pi d = \frac{1}{2}\pi(6) = 3\pi \text{ ft} = 9.4 \text{ ft}$.

Therefore, the total perimeter of the window is $6 + 12 + 12 + 9.4 = 39.4 \text{ ft}$. You need to buy 39.4 ft of trim.

PEOPLE IN MATHEMATICS

Archimedes

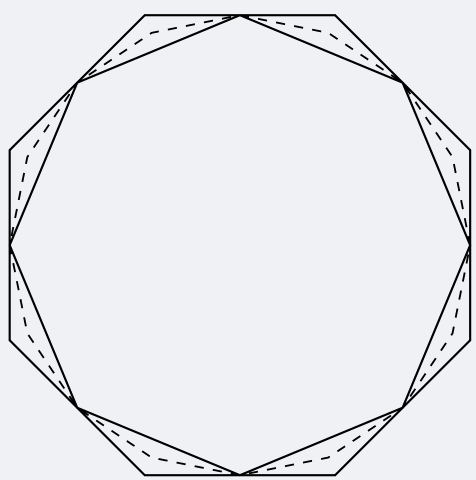
Figure 75: Archimedes

The overwhelming consensus is that Archimedes (287–212 BCE) was the greatest mathematician of classical antiquity, if not of all time. A Greek scientist, inventor, philosopher, astronomer, physicist, and mathematician, Archimedes flourished in Syracuse, Sicily. He is credited with the invention of various types of pulley systems and screw pumps based on the center of gravity. He advanced numerous mathematical concepts, including theorems for finding surface area and volume. Archimedes anticipated modern calculus and developed the idea of the "infinitely small" and the method of exhaustion. The method of exhaustion is a technique for finding the area of a shape inscribed within a sequence of polygons. The areas of the polygons converge to the area of the inscribed shape. This technique evolved to the concept of limits, which we use today.

One of the more interesting achievements of Archimedes is the way he estimated the number pi, the ratio of the circumference of a circle to its diameter. He was the first to find a valid approximation. He started with a circle having a diameter of 1 inch. His method involved drawing a polygon inscribed inside this circle and a polygon circumscribed around this circle. He knew that the perimeter of the inscribed polygon was smaller than the circumference of the circle, and the perimeter of the circumscribed polygon was larger than the circumference of the circle. This is shown in the drawing of an eight-sided polygon. He increased the number of sides of the polygon each time as he got closer to the real value of pi. The following table is an example of how he did this.

Sides	Inscribed Perimeter	Circumscribed Perimeter
4	2.8284	4.00
8	3.0615	3.3137
16	3.1214	3.1826
32	3.1365	3.1517
64	3.1403	3.1441

Concentric octagons and circle



Archimedes settled on an approximation of $\pi \approx 3.1416$ after an iteration of 96 sides. Because pi is an irrational number, it cannot be written exactly. However, the capability of the supercomputer can compute pi to billions of decimal digits. As of 2002, the most precise approximation of pi includes 1.2 trillion decimal digits.

WHO KNEW?

The Platonic Solids

The Platonic solids have been known since antiquity. A polyhedron is a three-dimensional object constructed with congruent regular polygonal faces. Named for the philosopher, Plato believed that each one of the solids is associated with one of the four elements: Fire is associated with the tetrahedron or pyramid, earth with the cube, air with the octahedron, and water with the icosahedron. Of the fifth Platonic solid, the dodecahedron, Plato said, "... God used it for arranging the constellations on the whole heaven."

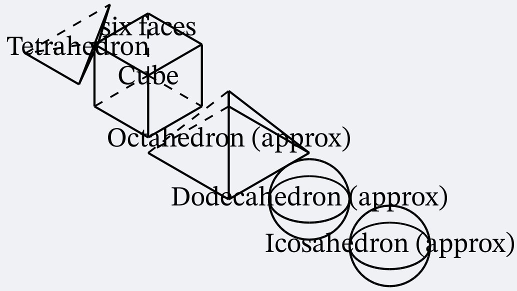


Figure 77: Platonic Solids

Plato believed that the combination of these five polyhedra formed all matter in the universe. Later, Euclid proved that exactly five regular polyhedra exist and devoted the last book of the *Elements* to this theory. These ideas were resuscitated by Johannes Kepler about 2,000 years later. Kepler used the solids to explain the geometry of the universe. The beauty and symmetry of the Platonic solids have inspired architects and artists from antiquity to the present.

Key Terms

- perimeter
- polygon
- pentagon
- hexagon
- heptagon
- octagon
- quadrilateral
- trapezoid
- parallelogram
- circumference

Key Concepts

- Regular polygons are closed, two-dimensional figures that have equal side lengths. They are named for the number of their sides.
- The perimeter of a polygon is the measure of the outline of the shape. We determine a shape's perimeter by calculating the sum of the lengths of its sides.

- The sum of the interior angles of a regular polygon with n sides is found using the formula $S = (n-2)180°$. The measure of a single interior angle of a regular polygon with n sides is determined using the formula $a = \frac{(n-2)180°}{n}$.
- The sum of the exterior angles of a regular polygon is 360°. The measure of a single exterior angle of a regular polygon with n sides is found using the formula $b = \frac{360°}{n}$.
- The circumference of a circle is $C = 2\pi r$, where r is the radius, or $C = \pi d$, and d is the diameter.

Formulas

The formula for the perimeter P of a rectangle is $P = 2L + 2W$, twice the length L plus twice the width W.

The sum of the interior angles of a polygon with n sides is given by

$$S = (n-2)180°.$$

The measure of each interior angle of a regular polygon with n sides is given by

$$a = \frac{(n-2)180°}{n}.$$

To find the measure of an exterior angle of a regular polygon with n sides we use the formula

$$b = \frac{360°}{n}.$$

The circumference of a circle is found using the formula $C = \pi d$, where d is the diameter of the circle, or $C = 2\pi r$, where r is the radius.

10.5 Tessellations

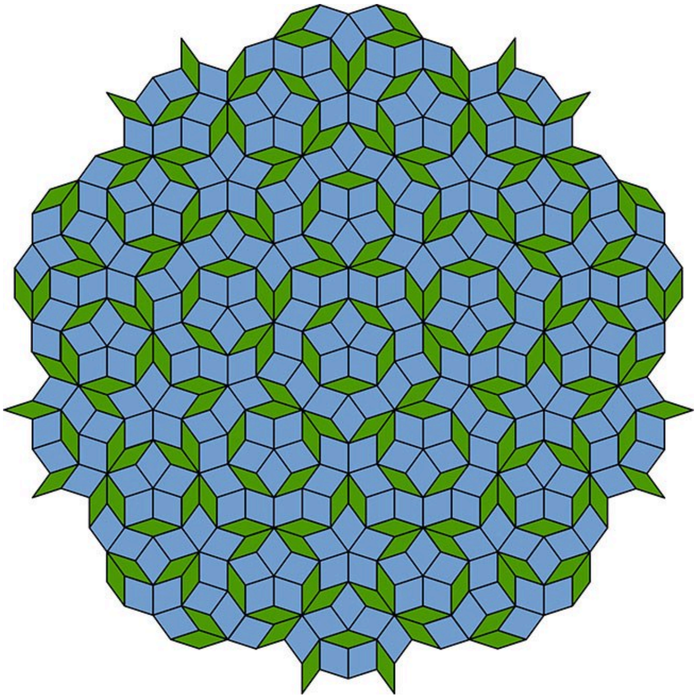


Figure 78: Penrose tiling represents one type of tessellation.

Learning Objectives

After completing this section, you should be able to:

1. Apply translations, rotations, and reflections.
2. Determine if a shape tessellates.

The illustration shown above is an unusual pattern called a Penrose tiling. Notice that there are two types of shapes used throughout the pattern: smaller green parallelograms and larger blue parallelograms. What's interesting about this design is that although it uses only two shapes over and over, there is no repeating pattern.

In this section, we will focus on patterns that do repeat. Repeated patterns are found in architecture, fabric, floor tiles, wall patterns, rug patterns, and many unexpected places as well. It may be a simple hexagon-shaped floor tile, or a complex pattern composed of several different motifs. These two-dimensional designs are called regular (or periodic) **tessellations**. There are countless designs that may be classified as regular tessellations, and they all have one thing in common—their patterns repeat and cover the plane.

We will explore how tessellations are created and experiment with making some of our own as well. The topic of tessellations belongs to a field in mathematics called transformational geometry, which is a study of the ways objects can be moved while retaining the same shape and size. These movements are termed *rigid motions* and *symmetries*.

WHO KNEW?

M. C. Escher

A good place to start the study of tessellations is with the work of M. C. Escher. The Dutch graphic artist was famous for the dimensional illusions he created in his woodcuts and lithographs, and that theme is carried out in many of his tessellations as well. Escher became obsessed with the idea of the "regular division of the plane." He sought ways to divide the plane with shapes that would fit snugly next to each other with no gaps or overlaps, represent beautiful patterns, and could be repeated infinitely to fill the plane. He experimented with practically every geometric shape imaginable and found the ones that would produce a regular division of the plane. The idea is similar to dividing a number by one of its factors. When a number divides another number evenly, there are no remainders, like there are no gaps when a shape divides or fills the plane.

Escher went far beyond geometric shapes, beyond triangles and polygons, beyond irregular polygons, and used other shapes like figures, faces, animals, fish, and practically any type of object to achieve his goal; and he did achieve it, beautifully, and left it for the ages to appreciate.

VIDEO

M.C. Escher: How to Create a Tessellation

The Mathematical Art of M.C. Escher

Tessellation Properties and Transformations

A regular tessellation means that the pattern is made up of congruent regular polygons, same size and shape, including *some type of movement* that is, some type of transformation or symmetry. Here we consider the rigid motions of translations, rotations, reflections, or glide reflections. A plane of tessellations has the following properties:

- Patterns are repeated and fill the plane.
- There are no gaps or overlaps. Shapes must fit together perfectly. (It was Escher who determined that a proper tessellation could have no gaps and no overlaps.)
- Shapes are combined using a transformation.
- All the shapes are joined at a vertex. In other words, if you were to draw a circle around a vertex, it would include a corner of each shape touching at that vertex.
- For a tessellation of regular congruent polygons, the sum of the measures of the interior angles that meet at a vertex equals 360°.

  In , the tessellation is made up of squares. There are four squares meeting at a vertex. An interior angle of a square is 90° and the sum of four interior angles is 360°. In , the tessellation is made up of regular hexagons. There are three hexagons meeting at each vertex. The interior angle of a hexagon is 120°, and the sum of three interior angles is 360°. Both tessellations will fill the plane, there are no gaps, the sum of the interior angle meeting at the vertex is 360°, and both are achieved by translation transformations. These tessellations work because all the properties of a tessellation are present.

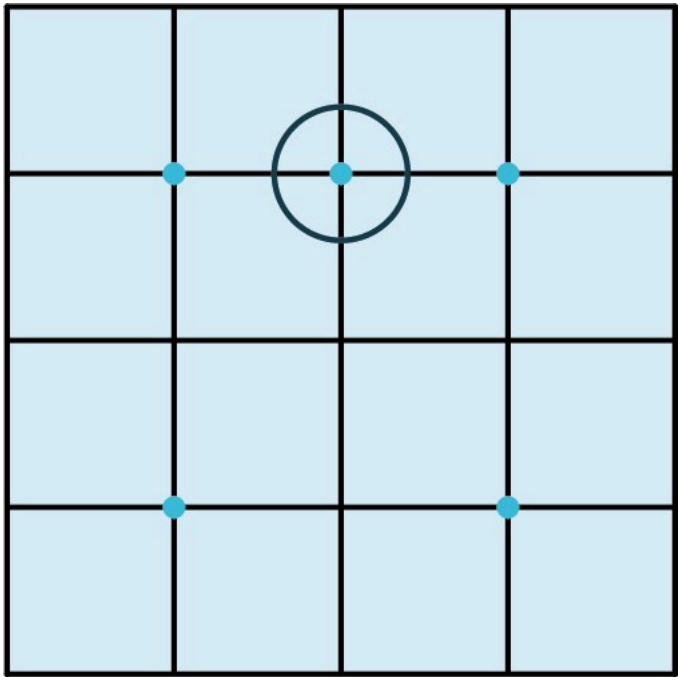


Figure 79: Tessellation – Squares

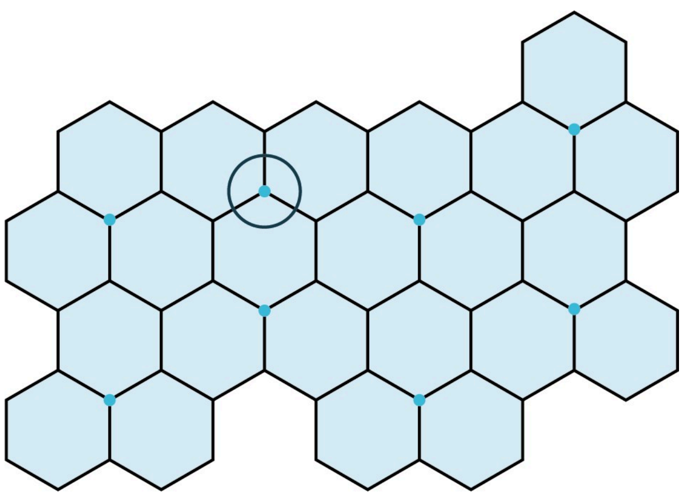


Figure 80: Tessellation – Hexagons

The movements or rigid motions of the shapes that define tessellations are classified as translations, rotations, reflections, or glide reflections. Let's first define these movements and then look at some examples showing how these transformations are revealed.

Translation

A **translation** is a movement that shifts the shape vertically, horizontally, or on the diagonal. Consider the trapezoid *ABCD* in . We have translated it 3 units to the right and 3 units up. That means every corner is moved by the number of units and in the direction specified. Mathematicians will indicate this movement with a vector, an arrow that is drawn to illustrate the criteria and the magnitude of the translation. The location of the translated trapezoid is marked with the vertices, $A'B'C'D'$, but it is still the exact same shape and size as the original trapezoid *ABCD*.

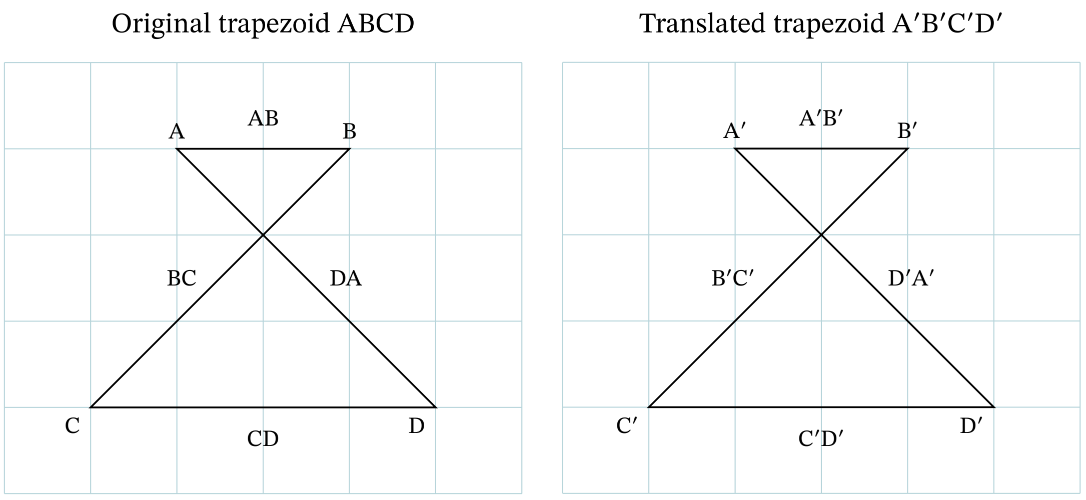
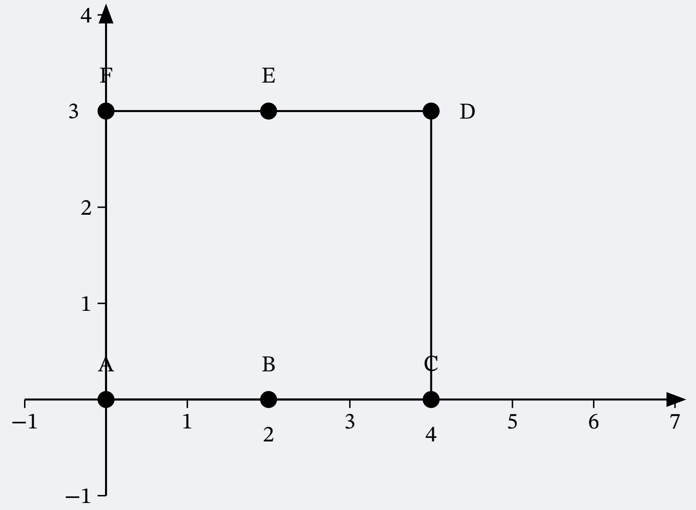


Figure 81: Translation

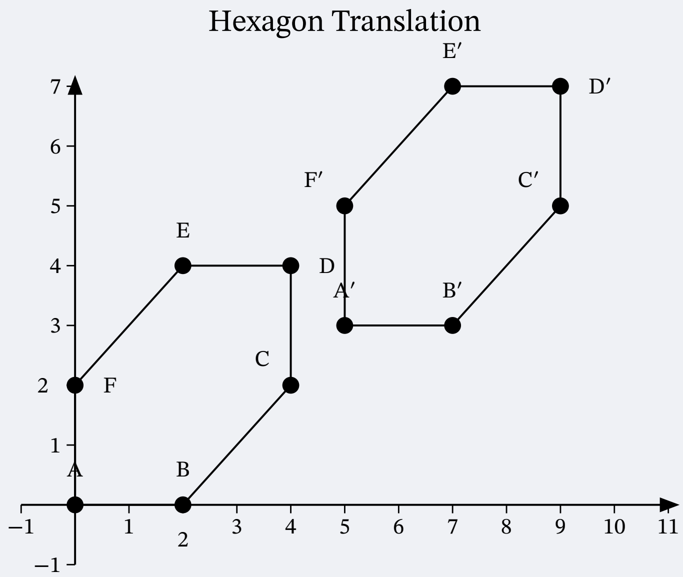
Example 1 **Creating a Translation**

Suppose you have a hexagon on a grid as in . Translate the hexagon 5 units to the right and 3 units up.



Solution

The best way to do this is by translating the individual points A, B, C, D, E, F. Once translated, the points become A', B', C', D', E', F'.



Rotation

The rotation transformation occurs when you rotate a shape about a point and at a predetermined angle. In , the triangle is rotated around the rotation point by 90°, and then translated 7 units up and 4 units over to the right. That means that each corner is translated to the new location by the same number of units and in the same direction.

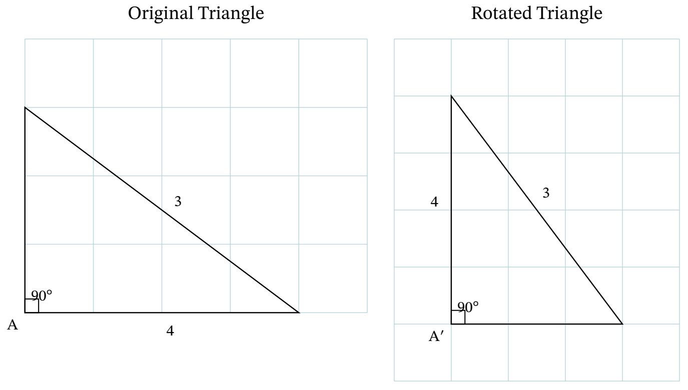


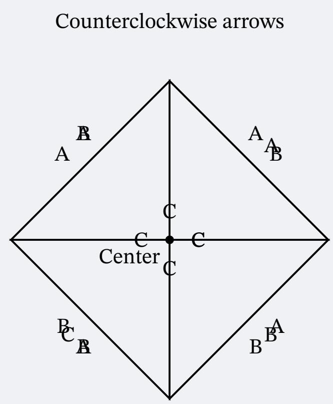
Figure 84: Rotation

We can see that ΔA is mapped to $\Delta A'$ by a rotation of 90° up and to the right. If rotated again by 90°, the triangle would be upside down.

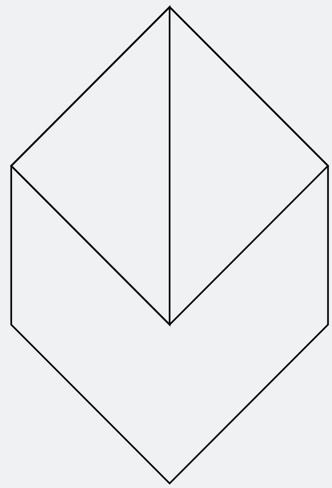
Example 2 **Applying a Rotation**

illustrates a tessellation begun with an equilateral triangle. Explain how this pattern is produced.

First tessellation pattern



Second tessellation pattern



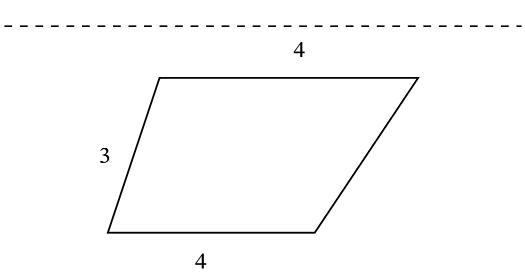
Solution

A rotation to the right or to the left around the vertex by 60°, six times, produces the hexagonal shape. The sixth rotation brings the triangle back to its original position. Then, a reflection up and another one on the diagonal will reproduce the pattern. When a shape returns to its original position by a rotation, we say that it has rotational symmetry.

Reflection

A **reflection** is the third transformation. A shape is reflected about a line and the new shape becomes a mirror image. You can reflect the shape vertically, horizontally, or on the diagonal. There are two shapes in . The quadrilateral is reflected horizontally; the arrow shape is reflected vertically.

Right trapezoid reflected vertically



Up arrow reflected horizontally

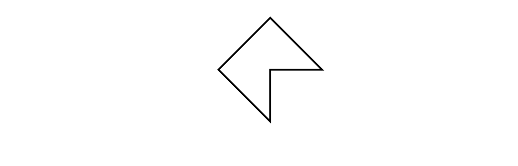


Figure 86: Reflection

Glide Reflection

The **glide reflection** is the fourth transformation. It is a combination of a reflection and a translation. This can occur by first reflecting the shape and then gliding or translating it to its new location, or by translating first and then reflecting. The example in shows a trapezoid, which is reflected over the dashed line, so it appears upside down. Then, we shifted the shape horizontally by 6 units to the right. Whether we use the glide first or the reflection first, the end result is the same in most cases. However, the tessellation shown in the next example can only be achieved by a reflection first and then a translation.

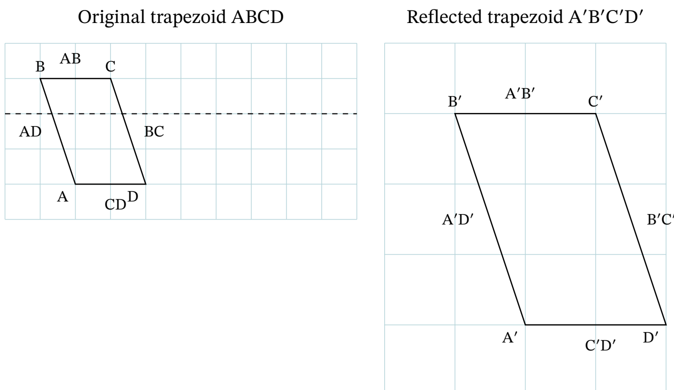
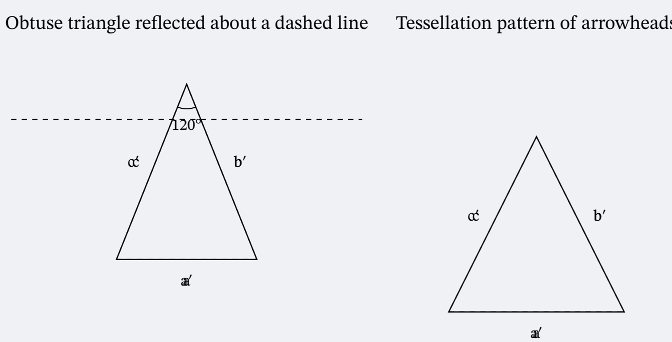


Figure 87: Glide Reflection

Example 3 **Applying the Glide Reflection**

An obtuse triangle is reflected about the dashed line, and the two shapes are joined together. How does the tessellation shown in materialize?



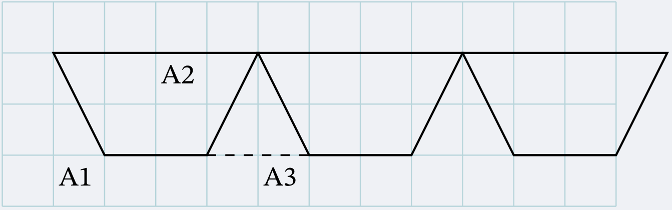
Solution

The new shape is reflected horizontally and joined with the original shape. It is then translated vertically and horizontally to make up the tessellation. Notice the blank spaces next to the vertical pattern. These areas are made up of the exact original shape rotated 180°, but with no line up the center. These rotated shapes are translated horizontally and vertically, and thus, the plane is tessellated with no gaps. This is an example of a glide reflection where the order of the transformations matters.

Example 4 **Applying More Than One Tessellation**

Show how this tessellation can be achieved.

Tessellation Pattern



Solution

This is a tessellation that has one color on the front of the trapezoid and a different color on the back. There is a translation on the diagonal, and a reflection vertically. These are two separate transformations resulting in two new placements of the trapezoid. We can call this a combination of two transformations or a glide reflection.

Interior Angles

The sum of the interior angles of a tessellation is 360°. In , the tessellation is made of six triangles formed into the shape of a hexagon. Each angle inside a triangle equals 60°, and the six vertices meet the sum of those interior angles, $6\left(60^{\circ}\right) = 360^{\circ}$.

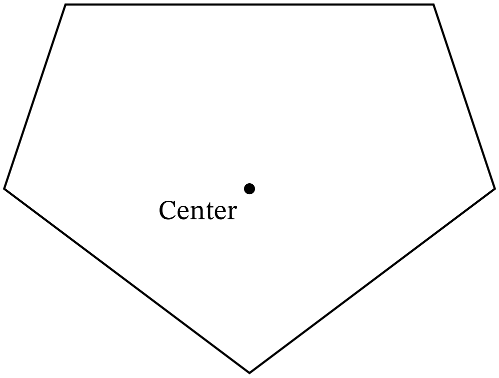


Figure 90: Interior Angles at the Vertex of Triangles

In , the tessellation is made up of trapezoids, such that two of the interior angles of each trapezoid equals 75° and the other two angles equal 105°. Thus, the sum of the interior angles where the vertices of four trapezoids meet equals $105^{\circ} + 75^{\circ} + 75^{\circ} + 105^{\circ} = 360^{\circ}$.

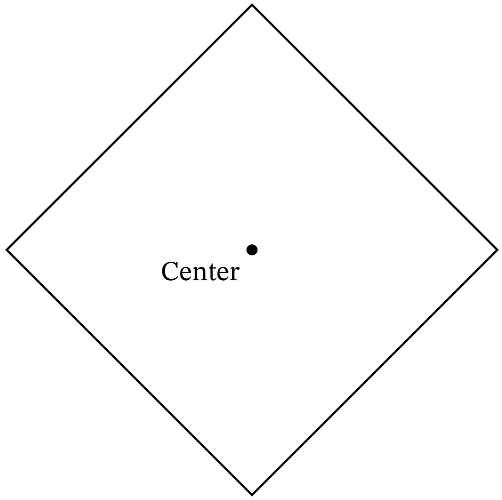


Figure 91: Interior Angles at the Vertex of Trapezoids

These tessellations illustrate the property that the shapes meet at a vertex where the interior angles sum to 360°.

Tessellating Shapes

We might think that all regular polygons will tessellate the plane by themselves. We have seen that squares do and hexagons do. The pattern of squares in is a translation of the shape horizontally and vertically. The hexagonal pattern in , is translated horizontally, and then on the diagonal, either to the right or to the left. This particular pattern can also be formed by rotations. Both tessellations are made up of congruent shapes and each shape fits in perfectly as the pattern repeats.

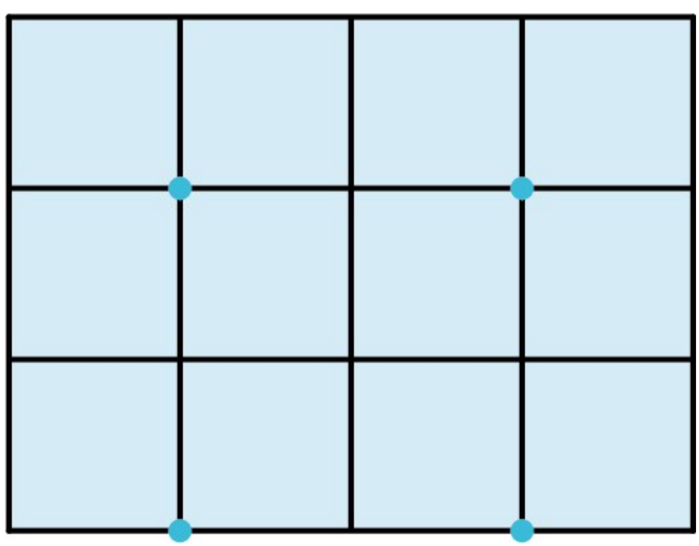


Figure 92: Translation Horizontally and Vertically

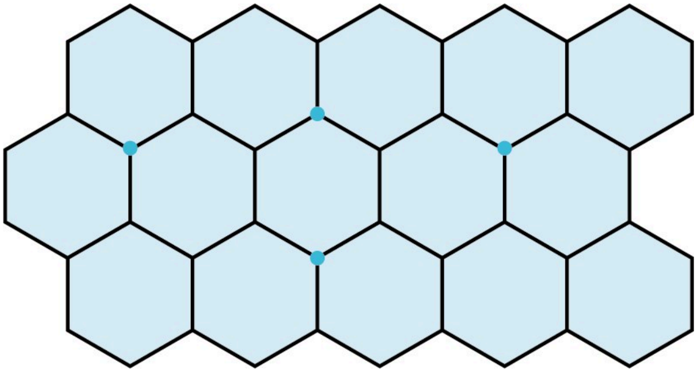


Figure 93: Translation Horizontally and Slide Diagonally

We have also seen that equilateral triangles will tessellate the plane without gaps or overlaps, as shown. The pattern is made by a reflection and a translation. The darker side is the face of the triangle and the lighter side is the back of the triangle, shown by the reflection. Each triangle is reflected and then translated on the diagonal.

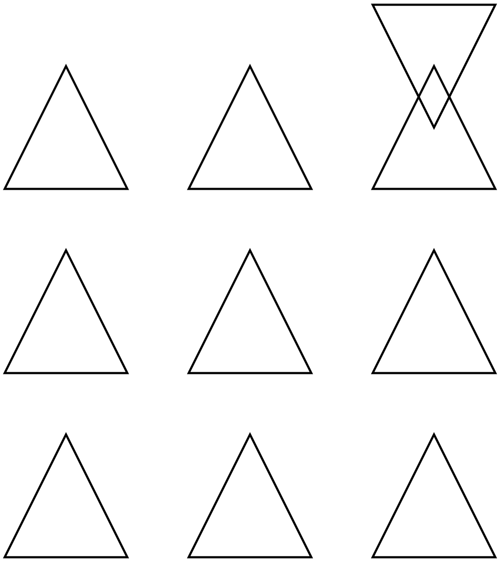
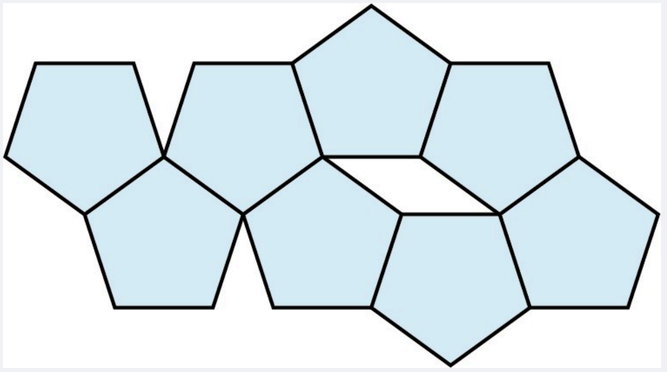


Figure 94: Reflection and Glide Translation

Escher experimented with all regular polygons and found that only the ones mentioned, the equilateral triangle, the square, and the hexagon, will tessellate the plane by themselves. Let's try a few other regular polygons to observe what Escher found.

Example 5 **Tessellating the Plane**

Do regular pentagons tessellate the plane by themselves ?

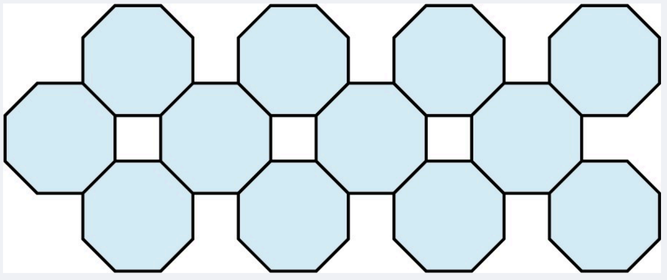


Solution

We can see that regular pentagons do not tessellate the plane by themselves. There is a gap, a gap in the shape of a parallelogram. We conclude that regular pentagons will not tessellate the plane by themselves.

Example 6 **Tessellating Octagons**

Do regular octagons tessellate the plane by themselves ?



Solution

Again, we see that regular octagons do not tessellate the plane by themselves. The gaps, however, are squares. So, two regular polygons, an octagon and a square, do tessellate the plane.

Just because regular pentagons do not tessellate the plane by themselves does not mean that there are no pentagons that tessellate the plane, as we see in .

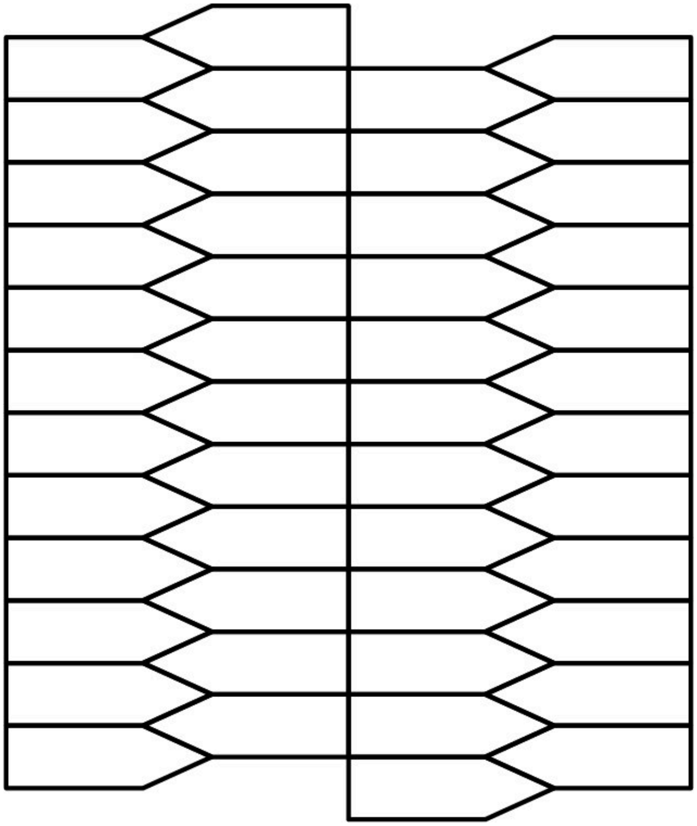


Figure 97: Tessellation of Pentagons

Another example of an irregular polygon that tessellates the plane is by using the obtuse irregular triangle from a previous example. What transformations should be performed to produce the tessellation shown in ?

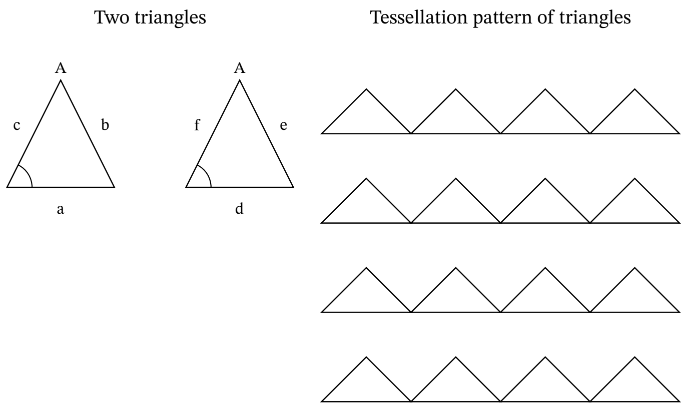


Figure 98: Tessellating with Obtuse Irregular Triangles

First, the triangle is reflected over the tip at point A, and then translated to the right and joined with the original triangle to form a parallelogram. The parallelogram is then translated on the diagonal and to the right and to the left.

Naming

A tessellation of squares is named by choosing a vertex and then counting the number of sides of each shape touching the vertex. Each square in the tessellation shown in has four sides, so starting with square A, the first number is 4, moving around counterclockwise to the next square meeting the vertex, square B, we have another 4, square C adds another 4, and finally square D adds a fourth 4. So, we would name this tessellation a 4.4.4.4.

The hexagon tessellation, shown in has six sides to the shape and three hexagons meet at the vertex. Thus, we would name this a 6.6.6. The triangle tessellation, shown in has six triangles meeting the vertex. Each triangle has three sides. Thus, we name this a 3.3.3.3.3.3.

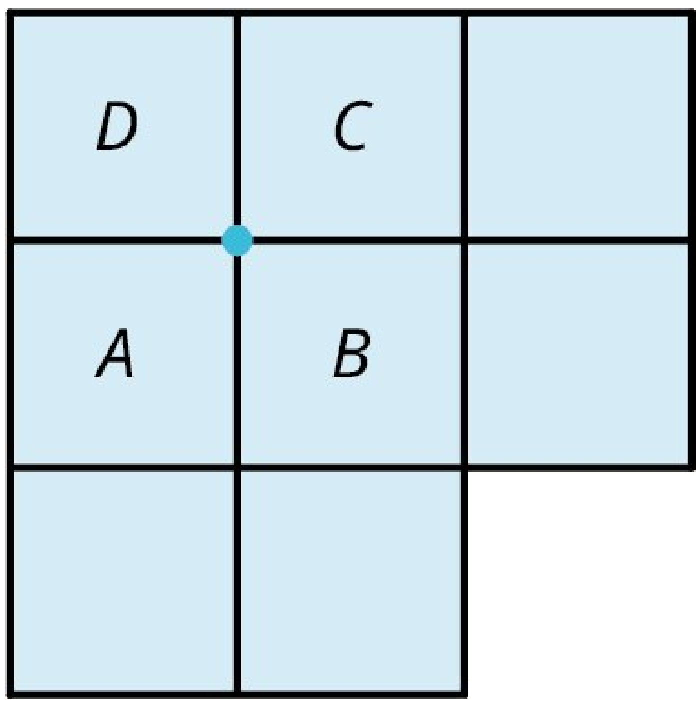


Figure 99: 4.4.4.4

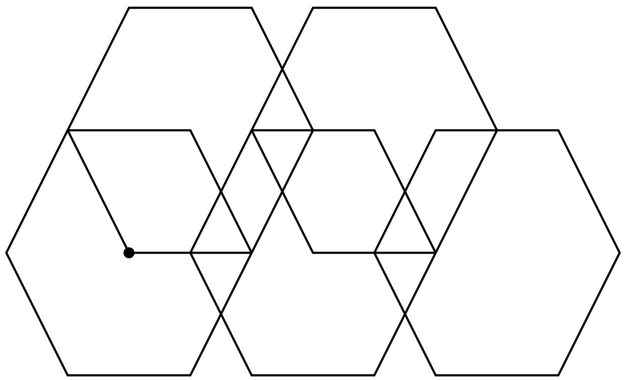


Figure 100: 6.6.6

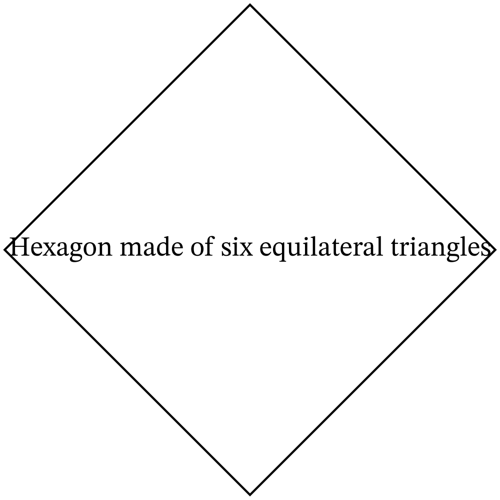


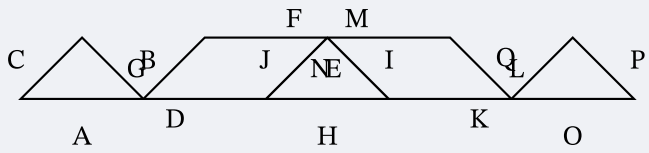
Figure 101: 3.3.3.3.3.3

Example 7 **Creating Your Own Tessellation**

Create a tessellation using two colors and two shapes.

Solution

We used a parallelogram and an isosceles triangle. The parallelogram is reflected vertically and horizontally so that only every other corner touches. The triangles are reflected vertically and horizontally and then translated over the parallelogram. The result is alternating vertical columns of parallelograms and then triangles .



Key Terms

- tessellation

- translation
- reflection
- rotation
- glide reflection

Key Concepts

- A tessellation is a particular pattern composed of shapes, usually polygons, that repeat and cover the plane with no gaps or overlaps.
- Properties of tessellations include rigid motions of the shapes called transformations. Transformations refer to translations, rotations, reflections, and glide reflections. Shapes are transformed in such a way to create a pattern.

Videos

- M.C. Escher: How to Create a Tessellation
- The Mathematical Art of M.C. Escher

10.6 Area

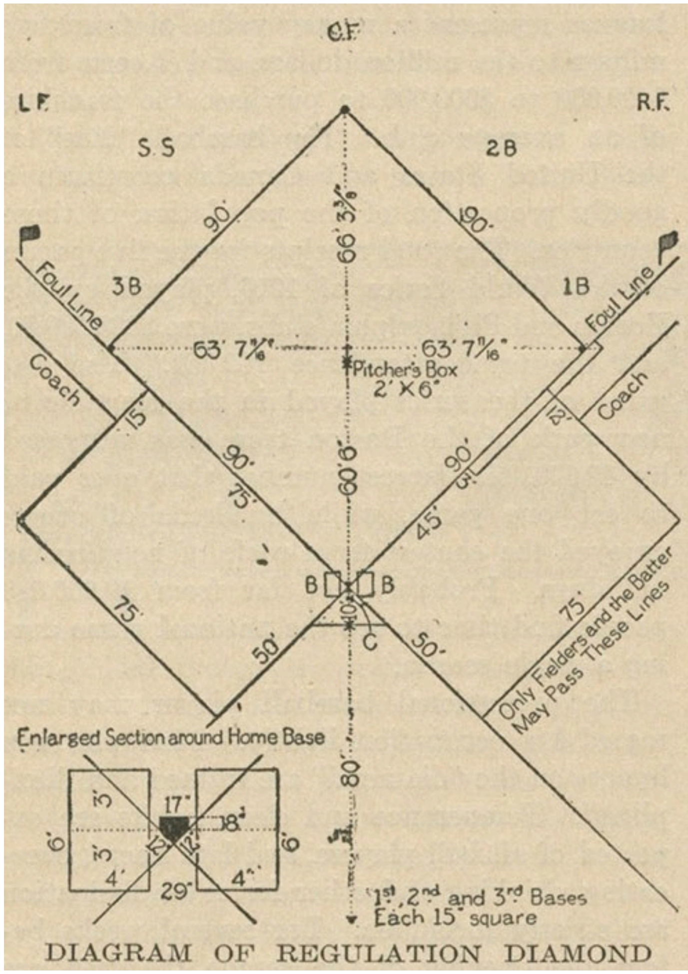


Figure 103: The area of a regulation baseball diamond must adhere to specific measurements to be legal.

Learning Objectives

After completing this section, you should be able to:

1. Calculate the area of triangles.
2. Calculate the area of quadrilaterals.
3. Calculate the area of other polygons.
4. Calculate the area of circles.

Some areas carry more importance than other areas. Did you know that in a baseball game, when the player hits the ball and runs to first base that he must run within a 6-foot wide path? If he veers off slightly to the right, he is out. In other words, a few inches can be the difference in winning or losing a game. Another example is real estate. On Manhattan Island, one square foot of real estate is worth far more than real estate in practically any other area of the country. In other words, we place a value on area. As the context changes, so does the value.

Area refers to a region measured in square units, like a square mile or a square foot. For example, to purchase tile for a kitchen floor, you would need to know how many square feet are needed because tile is sold by the square foot. Carpeting is sold by the square yard. As opposed to linear measurements like perimeter, which in in linear units. For example, fencing is sold in linear units, a linear foot or yard. Linear dimensions refer to an outline or a boundary. Square units refer to the area within that boundary. Different items may have different units, but either way, you must know the linear dimensions to calculate the area.

Many geometric shapes have formulas for calculating areas, such as triangles, regular polygons, and circles. To calculate areas for many irregular curves or shapes, we need calculus. However, in this section, we will only look at geometric shapes that have known area formulas. The notation for area, as mentioned, is in square units and we write sq in or sq cm, or use an exponent, such as in^2 or cm^2. Note that linear measurements have no exponent above the units or we can say that the exponent is 1.

Area of Triangles

The formula for the area of a triangle is given as follows.

> **FORMULA**
> The area of a triangle is given as $A = \frac{1}{2}bh$, where b represents the base and h represents the height.

For example, consider the triangle in .

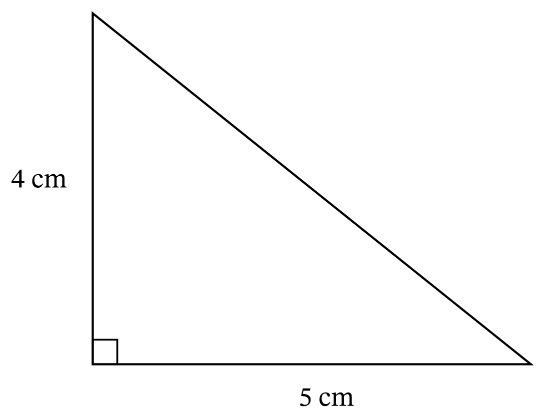


Figure 104: Triangle 1

The base measures 4 cm and the height measures 5 cm. Using the formula, we can calculate the area:

$$A = \frac{1}{2}(4)(5) = \frac{1}{2}(20) = 10 \text{ cm}^2$$

In , the triangle has a base equal to 7 cm and a height equal to 3.5 cm. Notice that we can only find the height by dropping a perpendicular to the base. The area is then

$$A = \frac{1}{2}(7)(3.5) = 12.25 \text{ cm}^2.$$

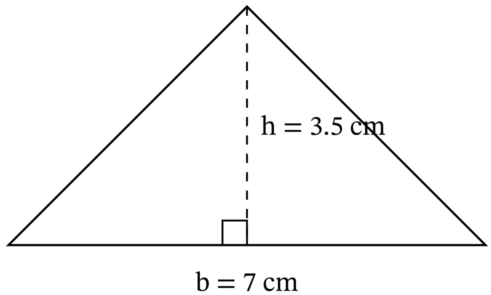
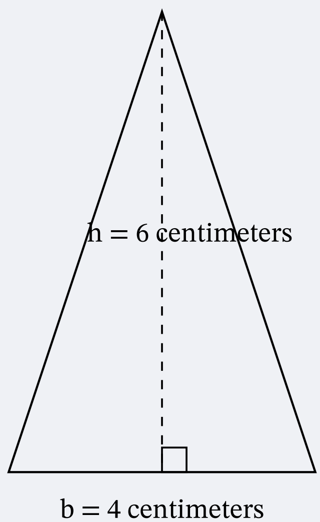


Figure 105: Triangle 2

Example 1 **Finding the Area of a Triangle**

Find the area of this triangle that has a base of 4 cm and the height is 6 cm .



Solution

Using the formula, we have

$$A = \frac{1}{2}(4)(6) = \frac{24}{2} = 12 \text{ cm}^2 .$$

Area of Quadrilaterals

A **quadrilateral** is a four-sided polygon with four vertices. Some quadrilaterals have either one or two sets of parallel sides. The set of quadrilaterals include the square, the rectangle, the parallelogram, the trapezoid, and the rhombus. The most common quadrilaterals are the square and the rectangle.

Square

In , a 12 in × 12 in grid is represented with twelve 1 in × 1 in squares across each row, and twelve 1 in × 1 in squares down each column. If you count the little squares, the sum equals 144 squares. Of course, you do not have to count little squares to find area—we have a formula. Thus, the formula for the area of a square, where s = length of a side, is $A = s \cdot s$. The area of the square in is $A = 12 \text{ in} \times 12 \text{ in} = 144 \text{ in}^2$.

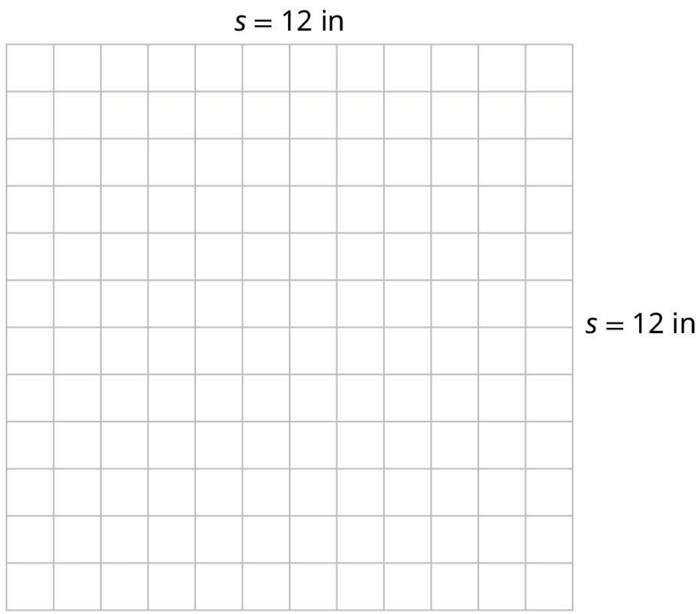


Figure 107: Area of a Square

> **FORMULA**
> The formula for the area of a square is $A = s \cdot s$ or $A = s^2$.

Rectangle

Similarly, the area for a rectangle is found by multiplying length times width. The rectangle in has width equal to 5 in and length equal to 12 in. The area is $A = 5(12) = 60 \text{ in}^2$.

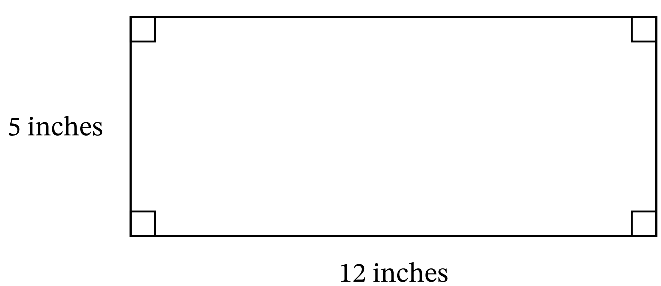


Figure 108: Area of a Rectangle

FORMULA

The area of a rectangle is given as $A = lw$.

Many everyday applications require the use of the perimeter and area formulas. Suppose you are remodeling your home and you want to replace all the flooring. You need to know how to calculate the area of the floor to purchase the correct amount of tile, or hardwood, or carpet. If you want to paint the rooms, you need to calculate the area of the walls and the ceiling to know how much paint to buy, and the list goes on. Can you think of other situations where you might need to calculate area?

Example 2 **Finding the Area of a Rectangle**

You have a garden with an area of 196 square feet. How wide can the garden be if the length is 28 feet?

Solution

The area of a rectangular region is $A = lw$. Letting the width equal w:

$$\begin{aligned} 196 &= 28w \\ \frac{196}{28} &= w = 7 \text{ ft} \end{aligned}$$

Example 3 **Determining the Cost of Floor Tile**

Jennifer is planning to install vinyl floor tile in her rectangular-shaped basement, which measures 29 ft by 16 ft. How many boxes of floor tile should she buy and how much will it cost if one box costs \$50 and covers 20 ft^2?

Solution

The area of the basement floor is $A = 29(16) = 464 \text{ ft}^2$. We will divide this area by 20 ft . Thus, $\frac{464}{20} = 23.2$. Therefore, Jennifer will have to buy 24 boxes of tile at a cost \$1,200.

Parallelogram

The area of a parallelogram can be found using the formula for the area of a triangle. Notice in , if we cut a diagonal across the parallelogram from one vertex to the opposite vertex, we have two triangles. If we multiply the area of a triangle by 2, we have the area of a parallelogram:

$$\begin{aligned} A &= 2\left(\frac{1}{2}bh\right) \\ A &= bh \end{aligned}$$

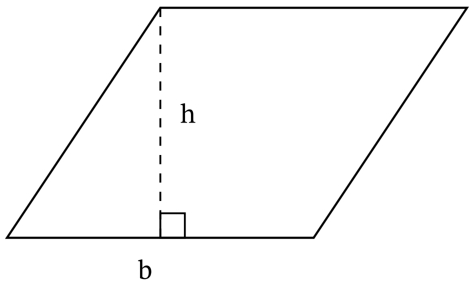
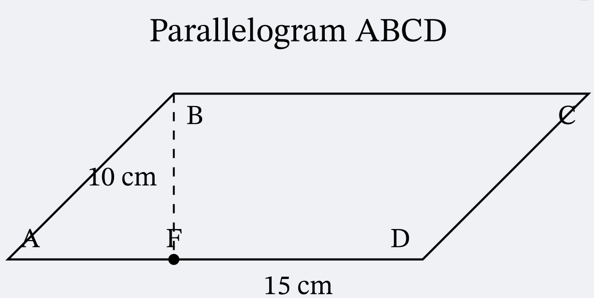


Figure 109: Area of a Parallelogram

FORMULA
The area of a parallelogram is $A = bh$.

For example, if we have a parallelogram with the base be equal to 10 inches and the height equal to 5 inches, the area will be $A = (10)(5) = 50 \text{ in}^2$.

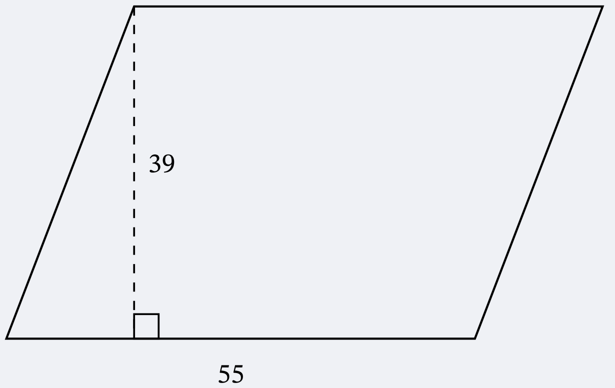
Example 4 Finding the Area of a Parallelogram
In the parallelogram , if $FB = 10, AD = 15$, find the exact area of the parallelogram.



Solution
Using the formula of $A = bh$, we have

$$A = 10(15) = 150 \text{ cm}^2.$$

Example 5 Finding the Area of a Parallelogram Park
The boundaries of a city park form a parallelogram . The park takes up one city block, which is contained by two sets of parallel streets. Each street measures 55 yd long. The perpendicular distance between streets is 39 yd. How much sod, sold by the square foot, should the city purchase to cover the entire park and how much will it cost? The sod is sold for $0.50 per square foot, installation is $1.50 per square foot, and the cost of the equipment for the day is $100.



Solution

Step 1: As sod is sold by the square foot, the first thing we have to do is translate the measurements of the park from yards to feet. There are 3 ft to a yard, so 55 yd is equal to 165 ft, and 39 yd is equal to 117 ft.

Step 2: The park has the shape of a parallelogram, and the formula for the area is $A = bh$:

$$A = 165(117) = 19,305 \text{ ft}^2.$$

Step 3: The city needs to purchase $19,305 \text{ ft}^2$ of sod. The cost will be $0.50 per square foot for the sod and $1.50 per square foot for installation, plus $100 for equipment:

$$\begin{aligned} 19,305(\$\,0.50) + 19,305(\$\,1.50) &= \$\,9,652.5 + \$\,28,957.5 + \$\,100.00 \\ &= \$\,38,710.00 \end{aligned}$$

Trapezoid

Another quadrilateral is the trapezoid. A trapezoid has one set of parallel sides or bases. The formula for the area of a trapezoid with parallel bases a and b and height h is given here.

FORMULA

The formula for the area of a trapezoid is given as $A = \frac{1}{2}h(a+b)$.

For example, find the area of the trapezoid in that has base a equal to 8 cm, base b equal to 6 cm, and height equal to 6 cm.

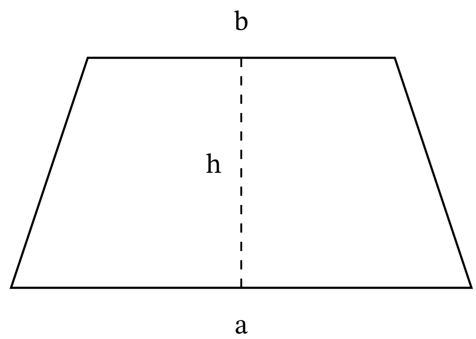
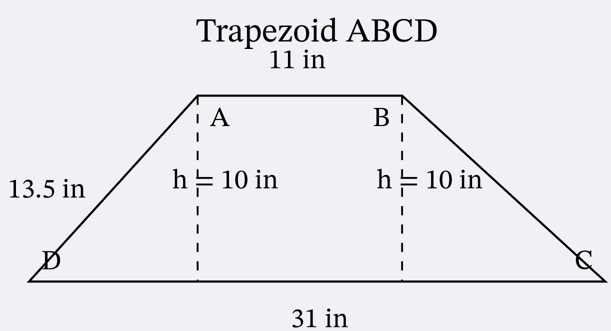


Figure 112: Area of a Trapezoid

The area is $A = \frac{1}{2}(6)(6 + 8) = 42 \text{ cm}^2$.

Example 6 **Finding the Area of a Trapezoid**

$ABCD$ is a regular trapezoid with $\overline{AB} \parallel \overline{CD}$. Find the exact perimeter of $ABCD$, and then find the area.



Solution

The perimeter is the measure of the boundary of the shape, so we just add up the lengths of the sides. We have $P = 31 + 13.5 + 11 + 13.5 = 69$ in . Then, the area of the trapezoid using the formula is $A = \frac{1}{2}(10)(11 + 31) = 210 \text{ in}^2$.

Rhombus

The rhombus has two sets of parallel sides. To find the area of a rhombus, there are two formulas we can use. One involves determining the measurement of the diagonals.

FORMULA

The area of a rhombus is found using one of these formulas:

- $A = \frac{d_1 d_2}{2}$, where d_1 and d_2 are the diagonals.
- $A = \frac{1}{2}bh$, where b is the base and h is the height.

For our purposes here, we will use the formula that uses diagonals. For example, if the area of a rhombus is 220 cm^2, and the measure of $d_2 = 11$, find the measure of d_1. To solve this problem, we input the known values into the formula and solve for the unknown.

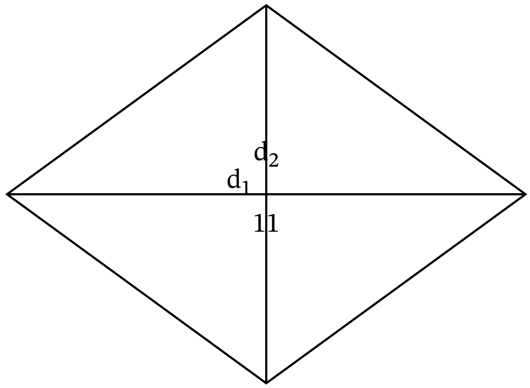


Figure 114: Area of a Rhombus

We have that

$$
\begin{aligned}
220 &= \frac{11d_1}{2} \\
220(2) &= 11d_1 \\
\frac{440}{11} &= d_1 = 40
\end{aligned}
$$

Example 7 **Finding the Area of a Rhombus**

Find the measurement of the diagonal d_1 if the area of the rhombus is 240 cm^2, and the measure of $d_2 = 24$ cm .

Solution

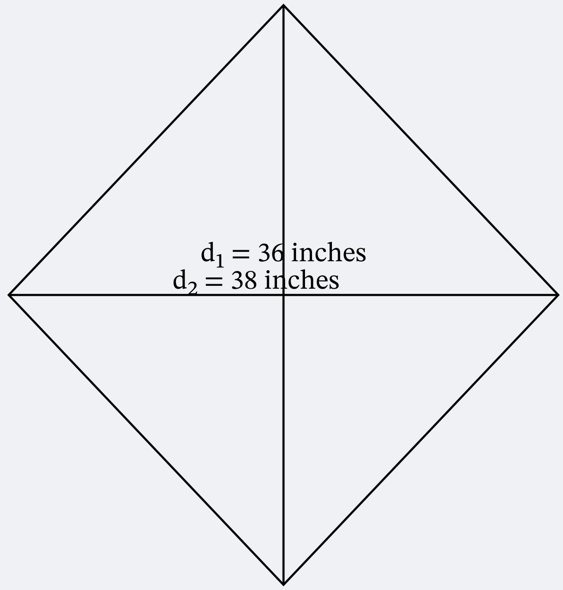
Use the formula with the known values:

$$
\begin{aligned}
240 &= \frac{d_1(24)}{2} \\
240(2) &= d_1(24) \\
\frac{480}{24} &= d_1 = 20
\end{aligned}
$$

Example 8 **Finding the Area of a Rhombus**

You notice a child flying a rhombus-shaped kite on the beach. When it falls to the ground, it falls on a beach towel measuring 36 in by 72 in . You notice that one of the diagonals of the kite is the same length as the 36 in width of the towel. The second diagonal appears to be 2 in longer. What is the area of the kite ?

Rhombus-shaped kite



Solution

Using the formula, we have:

$$A = \frac{d_1 d_2}{2}$$

$$= \frac{36(38)}{2} = 684\ \text{in}^2$$

Area of Polygons

To find the area of a regular polygon, we need to learn about a few more elements. First, the apothem a of a regular polygon is a line segment that starts at the center and is perpendicular to a side. The radius r of a regular polygon is also a line segment that starts at the center but extends to a vertex.

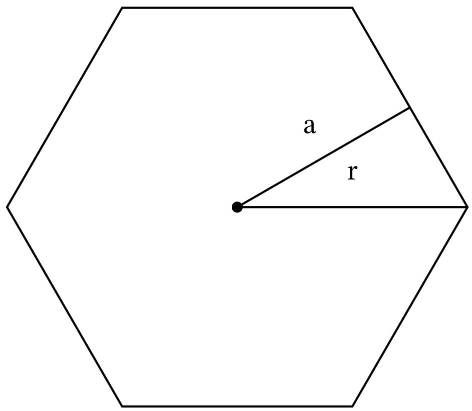


Figure 116: Apothem and Radius of a Polygon

FORMULA
The area of a regular polygon is found with the formula $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter.

For example, consider the regular hexagon shown in with a side length of 4 cm, and the apothem measures $a = 2\sqrt{3}$.

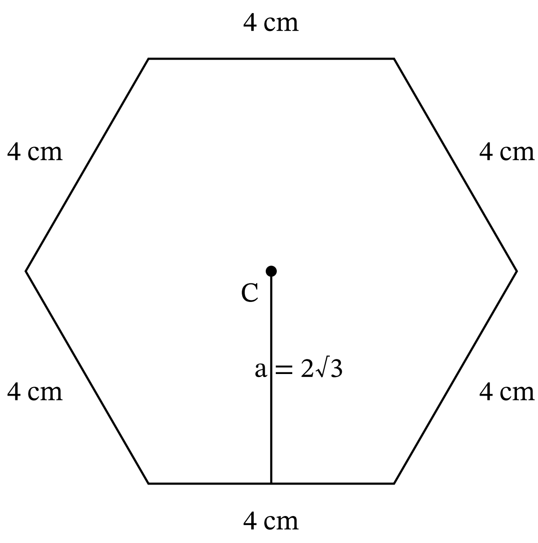


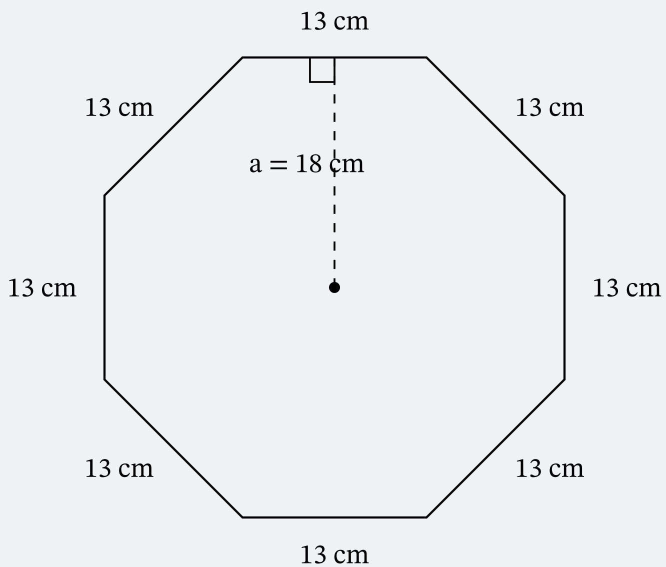
Figure 117: Area of a Hexagon

We have the perimeter, $p = 6(4) = 24$ cm . We have the apothem as $a = 2\sqrt{3}$. Then, the area is:

$$A = \frac{1}{2}\left(2\sqrt{3}\right)(24)$$
$$= 24\sqrt{3} = 41.57 \text{ cm}^2$$

Example 9 **Finding the Area of a Regular Octagon**

Find the area of a regular octagon with the apothem equal to 18 cm and a side length equal to 13 cm .



Solution

Using the formula, we have the perimeter $p = 8(13) = 104\text{ cm}$. Then, the area is $A = \frac{1}{2}(10)(104) = 936\text{ cm}^2$.

Changing Units

Often, we have the need to change the units of one or more items in a problem to find a solution. For example, suppose you are purchasing new carpet for a room measured in feet, but carpeting is sold in terms of yards. You will have to convert feet to yards to purchase the correct amount of carpeting. Or, you may need to convert centimeters to inches, or feet to meters. In each case, it is essential to use the correct equivalency.

Example 10 **Changing Units**

Carpeting comes in units of square yards. Your living room measures 21 ft wide by 24 ft long. How much carpeting do you buy?

Solution

We must convert feet to yards. As there are 3 ft in 1 yd, we have $21\text{ ft} = 7\text{ yds}$ and $24\text{ ft} = 8\text{ yds}$. Then, $7(8) = 56\text{ yd}^2$.

Area of Circles

Just as the circumference of a circle includes the number π, so does the formula for the area of a circle. Recall that π is a non-terminating, non-repeating decimal number: $\pi = 3.14159\ldots$.

It represents the ratio of the circumference to the diameter, so it is a critical number in the calculation of circumference and area.

FORMULA
The area of a circle is given as $A = \pi r^2$, where r is the radius.

For example, to find the of the circle with radius equal to 3 cm, as shown in , is found using the formula $A = \pi r^2$.

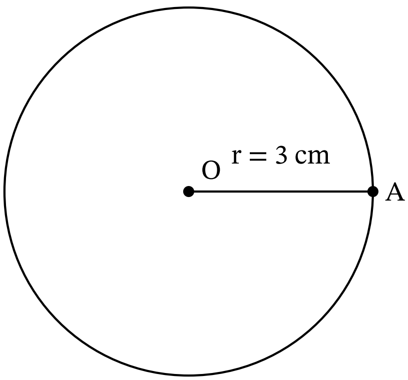


Figure 119: Circle with Radius 3

We have

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(3)^2 \\ &= 9\pi = 28.27\ \text{cm}^2 \end{aligned}$$

Example 11 **Finding the Area of a Circle**

Find the area of a circle with diameter of 16 cm.

Solution

The formula for the area of a circle is given in terms of the radius, so we cut the diameter in half. Then, the area is

$$A = \pi(8)^2 = 201.1\ \text{cm}^2\ .$$

Example 12 **Determining the Better Value for Pizza**

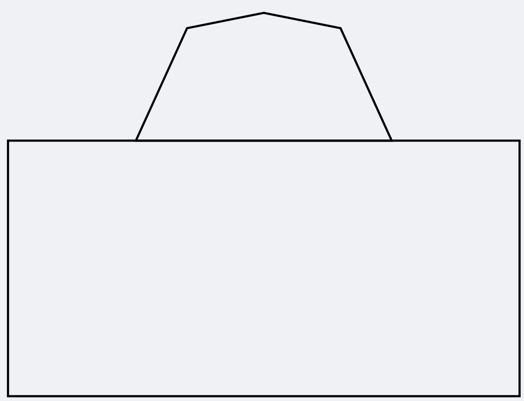
You decide to order a pizza to share with your friend for dinner. The price for an 8-inch diameter pizza is \$7.99. The price for 16-inch diameter pizza is \$13.99. Which one do you think is the better value?

Solution

The area of the 8-inch diameter pizza is $A = \pi(4)^2 = 50.3\ \text{in}^2$. The area of the 16-inch diameter pizza is $A = \pi(8)^2 = 201.1\ \text{in}^2$. Next, we divide the cost of each pizza by its area in square inches. Thus, $\frac{13.99}{201.1} = \$\,0.07$ per square inch and $\frac{7.99}{50.3} = \$\,0.16$ per square inch. So clearly, the 16-inch pizza is the better value.

Example 13 **Applying Area to the Real World**

You want to purchase a tinted film, sold by the square foot, for the window in . (This problem should look familiar as we saw it earlier when calculating circumference.) The bottom part of the window is a rectangle, and the top part is a semicircle. Find the area and calculate the amount of film to purchase.



Solution

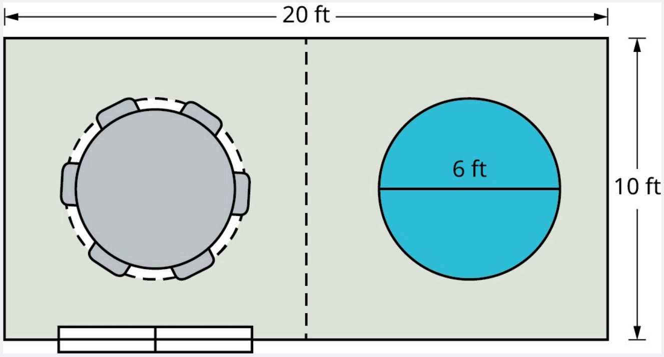
First, the rectangular portion has $A = 5(10) = 50\ \text{ft}^2$. For the top part, we have a semicircle with a diameter of 5 ft, so the radius is 2.5 ft. We want one half of the area of a circle with radius 2.5 ft, so the area of the top semicircle part is $A = \frac{1}{2}\pi(2.5)^2 = 9.8\ \text{ft}^2$. Add the area of the rectangle to the area of the semicircle. Then, the total area to be covered with the window film is $A = 50 + 9.8 = 59.8\ \text{ft}^2$.

Area within Area

Suppose you want to install a round hot tub on your backyard patio. How would you calculate the space needed for the hot tub? Or, let's say that you want to purchase a new dining room table, but you are not sure if you have enough space for it. These are common issues people face every day. So, let's take a look at how we solve these problems.

Example 14 **Finding the Area within an Area**

The patio in your backyard measures 20 ft by 10 ft . On one-half of the patio, you have a 4-foot diameter table with six chairs taking up an area of approximately 36 sq feet. On the other half of the patio, you want to install a hot tub measuring 6 ft in diameter. How much room will the table with six chairs and the hot tub take up? How much area is left over?



Solution

The hot tub has a radius of 3 ft. That area is then $A = \pi(3)^2 = 9\pi = 28.27\ \text{ft}^2$. The total square feet taken up with the table and chairs and the hot tub is $36 + 28.27 = 64.27\ \text{ft}^2$. The area left over is equal to the total area of the patio, $200\ \text{ft}^2$ minus the area for the table and chairs and the hot tub. Thus, the area left over is $200 - 64.27 = 135.7\ \text{ft}^2$.

Example 15 **Finding the Cost of Fertilizing an Area**

A sod farmer wants to fertilize a rectangular plot of land 150 ft by 240 ft. A bag of fertilizer covers $5,000\ \text{ft}^2$ and costs $200. How much will it cost to fertilize the entire plot of land?

Solution

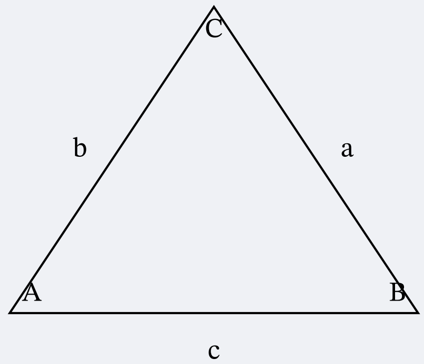
The plot of land is $36,000\ \text{ft}^2$. It will take 7.2 bags of fertilizer to cover the land area. Therefore, the farmer will have to purchase 8 bags of fertilizer at $200 a bag, which comes to $1,600.

PEOPLE IN MATHEMATICS

Heron of Alexandria

Figure 122: Heron of Alexandria

Heron of Alexandria, born around 20 A.D., was an inventor, a scientist, and an engineer. He is credited with the invention of the Aeolipile, one of the first steam engines centuries before the industrial revolution. Heron was the father of the vending machine. He talked about the idea of inserting a coin into a machine for it to perform a specific action in his book, *Mechanics and Optics*. His contribution to the field of mathematics was enormous. *Metrica*, a series of three books, was devoted to methods of finding the area and volume of three-dimensional objects like pyramids, cylinders, cones, and prisms. He also discovered and developed the procedures for finding square roots and cubic roots. However, he is probably best known for Heron's formula, which is used for finding the area of a triangle based on the lengths of its sides. Given a triangle ABC , triangle ABC



Heron's formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter calculated as $s = \frac{a+b+c}{2}$.

Key Terms

- triangle
- square
- rectangle
- rhombus
- apothem
- radius
- circle

Key Concepts

- The area A of a triangle is found with the formula $A = \frac{1}{2}bh$, where b is the base and h is the height.
- The area of a parallelogram is found using the formula $A = bh$, where b is the base and h is the height.
- The area of a rectangle is found using the formula $A = lw$, where l is the length and w is the width.
- The area of a trapezoid is found using the formula $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height, b_1 is the length of one base, and b_2 is the length of the other base.
- The area of a rhombus is found using the formula $A = \frac{d_1 d_2}{2}$, where d_1 is the length of one diagonal and d_2 is the length of the other diagonal.

- The area of a regular polygon is found using the formula $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter.
- The area of a circle is found using the formula $A = \pi r^2$, where r is the radius.

Formulas

The area of a triangle is given as $A = \frac{1}{2}bh$, where b represents the base and h represents the height.

The formula for the area of a square is $A = s \cdot s$ or $A = s^2$.

The area of a rectangle is given as $A = lw$.

The area of a parallelogram is $A = bh$.

The formula for the area of a trapezoid is given as $A = \frac{1}{2}h(a + b)$.

The area of a rhombus is found using one of these formulas:

- $A = \frac{d_1 d_2}{2}$, where d_1 and d_2 are the diagonals.
- $A = \frac{1}{2}bh$, where b is the base and h is the height.

  The area of a regular polygon is found with the formula $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter.

  The area of a circle is given as $A = \pi r^2$, where r is the radius.

10.7 Volume and Surface Area

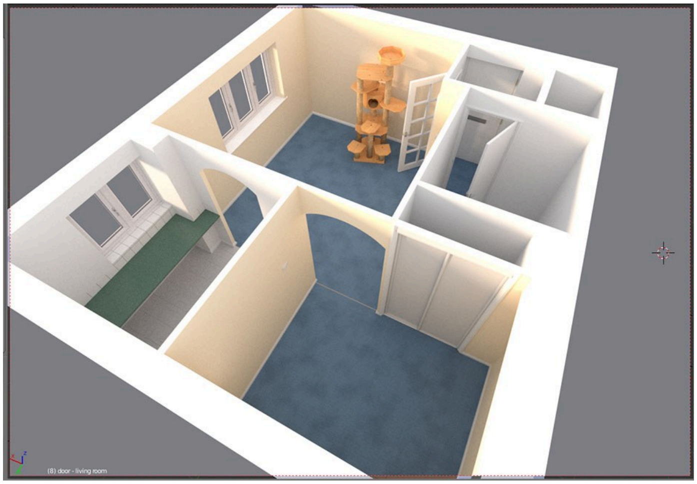


Figure 124: Volume is illustrated in this 3-dimensional view of an interior space. This gives a buyer a more realistic interpretation of space.

Learning Objectives

After completing this section, you should be able to:

1. Calculate the surface area of right prisms and cylinders.

2. Calculate the volume of right prisms and cylinders.
3. Solve application problems involving surface area and volume.

   Volume and surface area are two measurements that are part of our daily lives. We use volume every day, even though we do not focus on it. When you purchase groceries, volume is the key to pricing. Judging how much paint to buy or how many square feet of siding to purchase is based on surface area. The list goes on. An example is a three-dimensional rendering of a floor plan. These types of drawings make building layouts far easier to understand for the client. It allows the viewer a realistic idea of the product at completion; you can see the natural space, the volume of the rooms. This section gives you practical information you will use consistently. You may not remember every formula, but you will remember the concepts, and you will know where to go should you want to calculate volume or surface area in the future.

   We will concentrate on a few particular types of three-dimensional objects: right prisms and right cylinders. The adjective “right” refers to objects such that the sides form a right angle with the base. We will look at right rectangular prisms, right triangular prisms, right hexagonal prisms, right octagonal prisms, and right cylinders. Although, the principles learned here apply to all right prisms.

Three-Dimensional Objects

In geometry, three-dimensional objects are called **geometric solids**. Surface area refers to the flat surfaces that surround the solid and is measured in square units. Volume refers to the space inside the solid and is measured in cubic units. Imagine that you have a square flat surface with width and length. Adding the third dimension adds depth or height, depending on your viewpoint, and now you have a box. One way to view this concept is in the Cartesian coordinate three-dimensional space. The x-axis and the y-axis are, as you would expect, two dimensions and suitable for plotting two-dimensional graphs and shapes. Adding the z-axis, which shoots through the origin perpendicular to the xy-plane, and we have a third dimension.

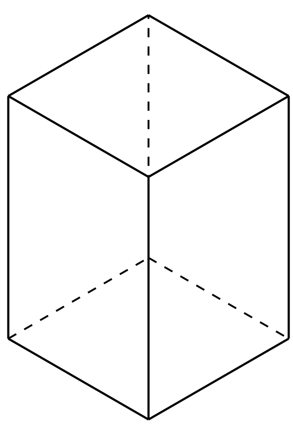


Figure 125: Three-Dimensional Space

Here is another view taking the two-dimensional square to a third dimension.

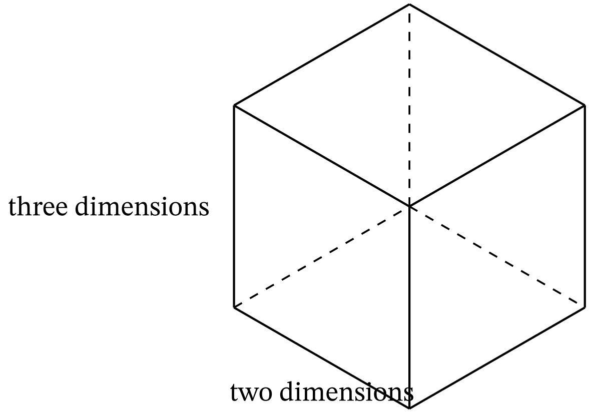
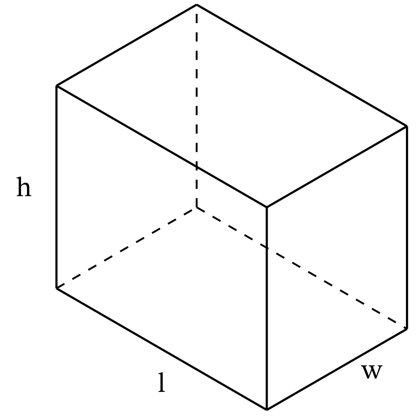


Figure 126: Going from Two Dimensions to Three Dimensions

To study objects in three dimensions, we need to consider the formulas for surface area and volume. For example, suppose you have a box with a hinged lid that you want to use for keeping photos, and you want to cover the box with a decorative paper. You would need to find the surface area to calculate how much paper is needed. Suppose you need to know how much water will fill your swimming pool. In that case, you would need to calculate the volume of the pool. These are just a couple of examples of why these concepts should be understood, and familiarity with the formulas will allow you to make use of these ideas as related to right prisms and right cylinders.



Right Prisms

A **right prism** is a particular type of three-dimensional object. It has a polygon-shaped base and a congruent, regular polygon-shaped top, which are connected by the height of its lateral sides, as shown. The lateral sides form a right angle with the base and the top. There are rectangular prisms, hexagonal prisms, octagonal prisms, triangular prisms, and so on.

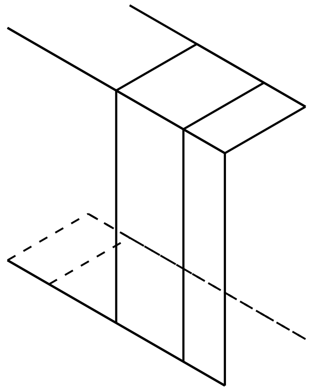


Figure 128: Pentagonal Prism

Generally, to calculate surface area, we find the area of each side of the object and add the areas together. To calculate volume, we calculate the space inside the solid bounded by its sides.

FORMULA

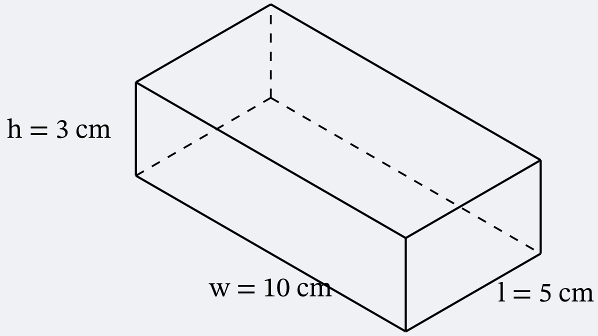
The formula for the **surface area** of a right prism is equal to twice the area of the base plus the perimeter of the base times the height, $SA = 2B + ph$, where B is equal to the area of the base and top, p is the perimeter of the base, and h is the height.

FORMULA

The formula for the volume of a rectangular prism, given in cubic units, is equal to the area of the base times the height, $V = B \cdot h$, where B is the area of the base and h is the height.

Example 1 **Calculating Surface Area and Volume of a Rectangular Prism**

Find the surface area and volume of the rectangular prism that has a width of 10 cm, a length of 5 cm, and a height of 3 cm .



Solution

The surface area is $SA = 2(10)(5) + 2(5)(3) + 2(10)(3) = 190\ \text{cm}^2$.

The volume is $V = 10(5)(3) = 150\ \text{cm}^3$.

In , we have three views of a right hexagonal prism. The regular hexagon is the base and top, and the lateral faces are the rectangular regions perpendicular to the base. We call it a right prism because the angle formed by the lateral sides to the base is 90°.

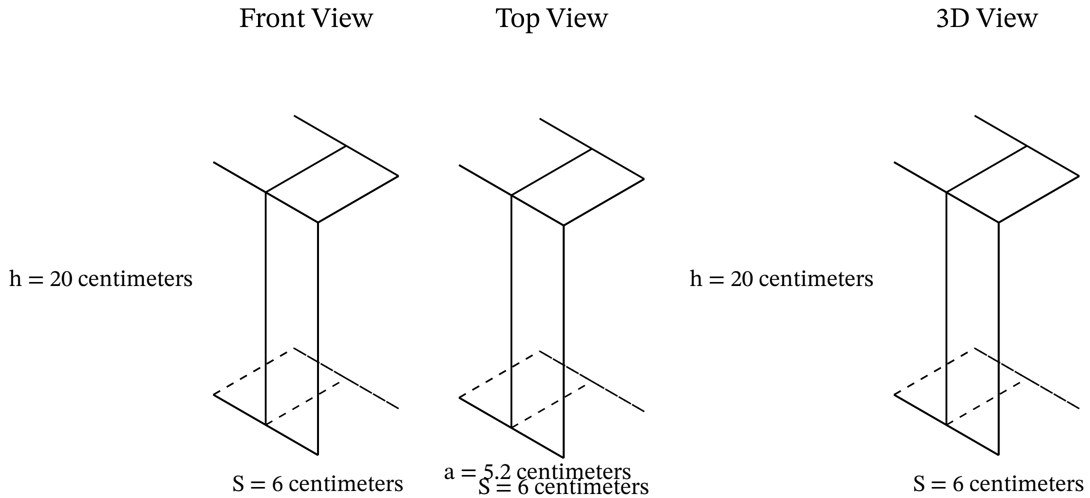


Figure 130: Right Hexagonal Prism

The first image is a view of the figure straight on with no rotation in any direction. The middle figure is the base or the top. The last figure shows you the solid in three dimensions. To calculate the surface area of the right prism shown in , we first determine the area of the hexagonal base and multiply that by 2, and then add the perimeter of the base times the height. Recall the area of a regular polygon is given as $A = \frac{1}{2}ap$, where a is the apothem and p is the perimeter. We have that

$$A_{base} = \frac{1}{2}(5.2)(36) = 93.6 \text{ cm}^2$$

Then, the surface area of the hexagonal prism is

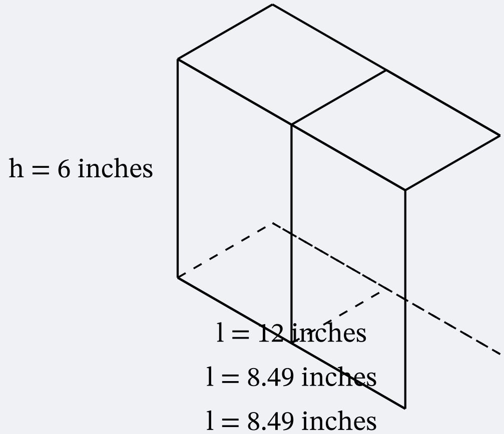
$$SA = 2(93.6) + 36(20) = 907.2 \text{ in}^2$$

To find the volume of the right hexagonal prism, we multiply the area of the base by the height using the formula $V = Bh$. The base is 93.6 cm^2, and the height is 20 cm. Thus,

$$V = 93.6(20) = 1872 \text{ cm}^3.$$

Example 2 **Calculating the Surface Area of a Right Triangular Prism**

Find the surface area of the triangular prism .

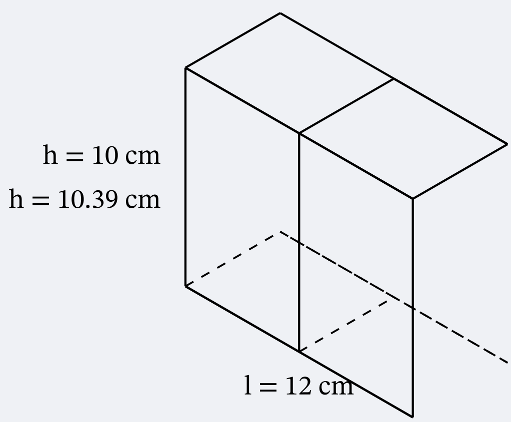


Solution

The area of the triangular base is $A_{base} = \frac{1}{2}(12)(6) = 36\ \text{in}^2$. The perimeter of the base is $p = 12 + 8.49 + 8.49 = 28.98$ in . Then, the surface area of the triangular prism is $SA = 2(36) + 28.98(10) = 361.8\ \text{in}^2$.

Example 3 **Finding the Surface Area and Volume**

Find the surface area and the volume of the right triangular prism with an equilateral triangle as the base and height .



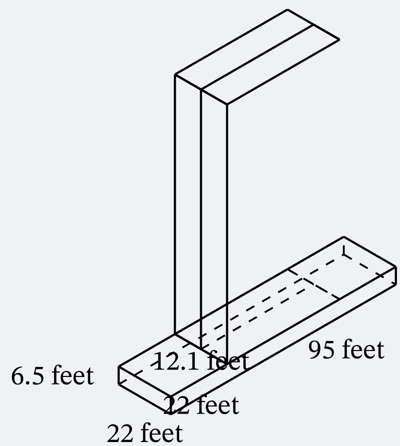
Solution

The area of the triangular base is $A_{base} = \frac{1}{2}(6)(10.39) = 31.17\ \text{cm}^2$. Then, the surface area is $SA = 2(31.17) + 36(10) = 422.34\ \text{cm}^2$.

The volume formula is found by multiplying the area of the base by the height. We have that $V = B \cdot h = (31.17)(10) = 311.7\ \text{cm}^3$.

Example 4 **Determining Surface Area Application**

Katherine and Romano built a greenhouse made of glass with a metal roof . In order to determine the heating and cooling requirements, the surface area must be known. Calculate the total surface area of the greenhouse.



Solution

The area of the long side measures $95 \text{ ft} \times 6.5 \text{ ft} = 1,235 \text{ ft}^2$. Multiplying by 2 gives $2,470 \text{ ft}^2$. The front (minus the triangular area) measures $22 \text{ ft} \times 6.5 \text{ ft} = 286 \text{ ft}^2$. Multiplying by 2 gives 572 ft^2. The floor measures $95 \text{ ft} \times 22 \text{ ft} = 2,090 \text{ ft}^2$. Each triangular region measures $A = \frac{1}{2}(22)(5) = 55 \text{ ft}^2$. Multiplying by 2 gives 110 ft^2. Finally, one side of the roof measures $12.1 \text{ ft} \times 95 \text{ ft} = 1,149.5 \text{ ft}^2$. Multiplying by 2 gives 2299 ft^2. Add them up and we have $SA = 2,470 + 572 + 2,090 + 110 + 2,299 = 7,541 \text{ ft}^2$.

Right Cylinders

There are similarities between a prism and a cylinder. While a prism has parallel congruent polygons as the top and the base, a **right cylinder** is a three-dimensional object with congruent circles as the top and the base. The lateral sides of a right prism make a 90° angle with the polygonal base, and the side of a cylinder, which unwraps as a rectangle, makes a 90° angle with the circular base.

Right cylinders are very common in everyday life. Think about soup cans, juice cans, soft drink cans, pipes, air hoses, and the list goes on.

In , imagine that the cylinder is cut down the 12-inch side and rolled out. We can see that the cylinder side when flat forms a rectangle. The SA formula includes the area of the circular base, the circular top, and the area of the rectangular side. The length of the rectangular side is the circumference of the circular base. Thus, we have the formula for total surface area of a right cylinder.

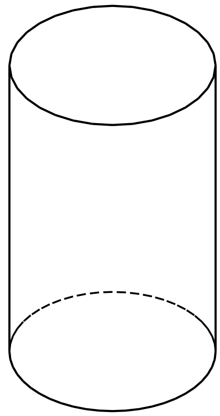


Figure 134: Right Cylinder

FORMULA

The surface area of a right cylinder is given as $SA = 2\pi r^2 + 2\pi rh$.

To find the volume of the cylinder, we multiply the area of the base with the height.

FORMULA

The volume of a right cylinder is given as $V = \pi r^2 h$.

Example 5 **Finding the Surface Area and Volume of a Cylinder**

Given the cylinder in , which has a radius of 5 inches and a height of 12 inches, find the surface area and the volume.

Solution

Step 1: We begin with the areas of the base and the top. The area of the circular base is

$$A_{base} = \pi(5)^2 = 25\pi = 78.5 \text{ in}^2.$$

Step 2: The base and the top are congruent parallel planes. Therefore, the area for the base and the top is

$$A = 2(78.5) = 157 \text{ in}^2.$$

Step 3: The area of the rectangular side is equal to the circumference of the base times the height:

$$A = 2\pi(5)(12) = 377 \text{ in}^2$$

Step 4: We add the area of the side to the areas of the base and the top to get the total surface area. We have

$$SA = 157 + 377 = 534 \text{ in}^2$$

Step 5: The volume is equal to the area of the base times the height. Then,

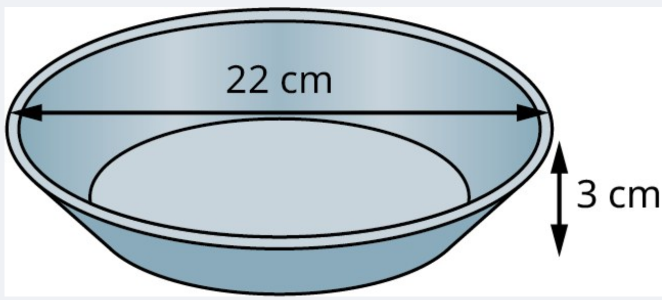
$$V = \pi(5)^2(12) = 942.48 \text{ in}^3.$$

Applications of Surface Area and Volume

The following are just a small handful of the types of applications in which surface area and volume are critical factors. Give this a little thought and you will realize many more practical uses for these procedures.

Example 6 **Applying a Calculation of Volume**

A can of apple pie filling has a radius of 4 cm and a height of 10 cm. How many cans are needed to fill a pie pan measuring 22 cm in diameter and 3 cm deep?



Solution

The volume of the can of apple pie filling is $V = \pi(4)^2(10) = 502.7 \text{ cm}^3$. The volume of the pan is $V = \pi(11)^2(3) = 1140.4 \text{ cm}^3$. To find the number of cans of apple pie filling, we divide the volume of the pan by the volume of a can of apple pie filling. Thus, $1140 \div 502.7 = 2.3$. We will need 2.3 cans of apple pie filling to fill the pan.

Optimization

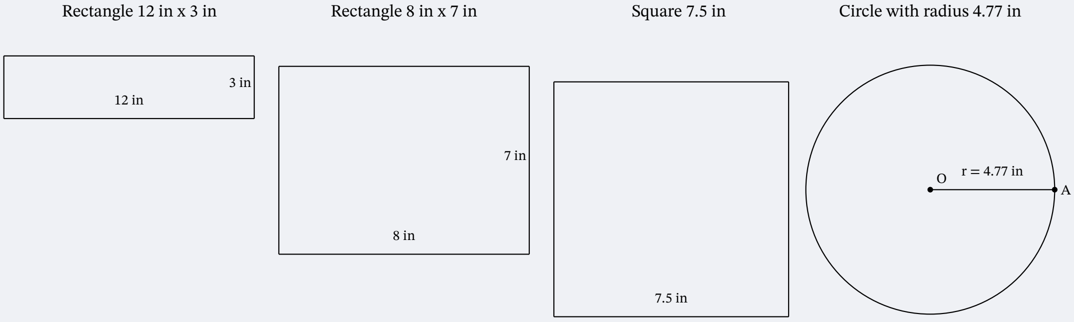
Problems that involve optimization are ones that look for the best solution to a situation under some given conditions. Generally, one looks to calculus to solve these problems. However, many geometric applications can be solved with the tools learned in this section. Suppose you want to make some throw pillows for your sofa, but you have a limited amount of fabric. You want to make the largest pillows you can from the fabric you have, so you would need to figure out the dimensions of the pillows that will fit these criteria. Another situation might be that you want to fence off an area in your backyard for a garden. You have a limited amount of fencing available, but you would like the garden to be as large as possible. How would you determine the shape and size of the garden? Perhaps you are looking for maximum volume or minimum surface area. Minimum cost is also a popular application of optimization. Let's explore a few examples.

Example 7 **Maximizing Area**

Suppose you have 150 meters of fencing that you plan to use for the enclosure of a corral on a ranch. What shape would give the greatest possible area?

Solution

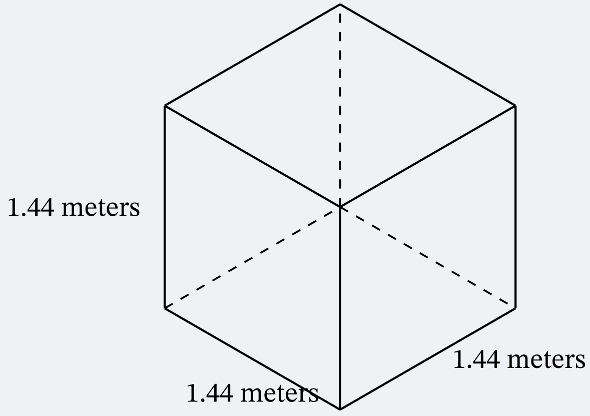
So, how would we start? Let's look at this on a smaller scale. Say that you have 30 inches of string and you experiment with different shapes. The rectangle in measures 12 in long by 3 in wide. We have a perimeter of $P = 2(12) + 2(3) = 30$ in and the area calculates as $A = 3(12) = 36\text{ in}^2$. The rectangle in , measures 8 in long and 7 in wide. The perimeter is $P = 2(8) + 2(7) = 30$ in and the area is $A = 8(7) = 56\text{ in}^2$. In , the square measures 7.5 in on each side. The perimeter is then $P = 4(7.5) = 30$ in and the area is $A = 7.5(7.5) = 56.25\text{ in}^2$. If you want a circular corral as in , we would consider a circumference of $30 = 2\pi r$, which gives a radius of $30 \div 2\pi = 4.77$ in and an area of $A = \pi(4.77)^2 = 71.5\text{ in}^2$.



We see that the circular shape gives the maximum area relative to a circumference of 30 in. So, a circular corral with a circumference of 150 meters and a radius of 23.87 meters gives a maximum area of $1,790.5\text{ m}^2$.

Example 8 **Designing for Cost**

Suppose you want to design a crate built out of wood in the shape of a rectangular prism . It must have a volume of 3 cubic meters. The cost of wood is $15 per square meter. What dimensions give the most economical design while providing the necessary volume?



Solution

To choose the optimal shape for this container, you can start by experimenting with different sizes of boxes that will hold 3 cubic meters. It turns out that, similar to the maximum rectangular area example where a square gives the maximum area, a cube gives the maximum volume and the minimum surface area.

As all six sides are the same, we can use a simplified volume formula:

$$V = s^3,$$

where s is the length of a side. Then, to find the length of a side, we take the cube root of the volume.

We have

$$\begin{aligned} 3 &= s^3 \\ \sqrt[3]{3} &= s \\ &= 1.4422 \text{ m} \end{aligned}$$

The surface area is equal to the sum of the areas of the six sides. The area of one side is $A = (1.4422)^2 = 2.08 \text{ m}^2$. So, the surface area of the crate is $SA = 6(2.08) = 12.5 \text{ m}^2$. At $15 a square meter, the cost comes to $12.5(\$\,15) = \$\,187.50$. Checking the volume, we have $V = (1.4422)^3 = 2.99 \text{ m}^3$.

Key Terms

- surface area
- volume
- right prism
- right cylinder

Key Concepts

- A right prism is a three-dimensional object that has a regular polygonal face and congruent base such that that lateral sides form a 90° angle with the base and top. The surface area SA of a right prism is found using the formula $SA = 2B + ph$, where B is the area of the base, p is the perimeter of the base, and h is the height. The volume V of a right prism is found using the formula $V = Bh$, where B is the area of the base and h is the height.
- A right cylinder is a three-dimensional object with a circle as the top and a congruent circle is the base, and the side forms a 90° angle to the base and top. The surface area of a right cylinder is found using the formula $SA = 2\pi r^2 + 2\pi rh$, where r is the radius and h is the height. The volume is found using the formula $V = \pi r^2 h$, where r is the radius and h is the height.

Formulas

The formula for the surface area of a right prism is equal to twice the area of the base plus the perimeter of the base times the height, $SA = 2B + ph$, where B is equal to the area of the base and top, p is the perimeter of the base, and h is the height.

The formula for the volume of a rectangular prism, given in cubic units, is equal to the area of the base times the height, $V = B \cdot h$, where B is the area of the base and h is the height.

The surface area of a right cylinder is given as $SA = 2\pi r^2 + 2\pi rh$.

The volume of a right cylinder is given as $V = \pi r^2 h$.

10.8 Right Triangle Trigonometry

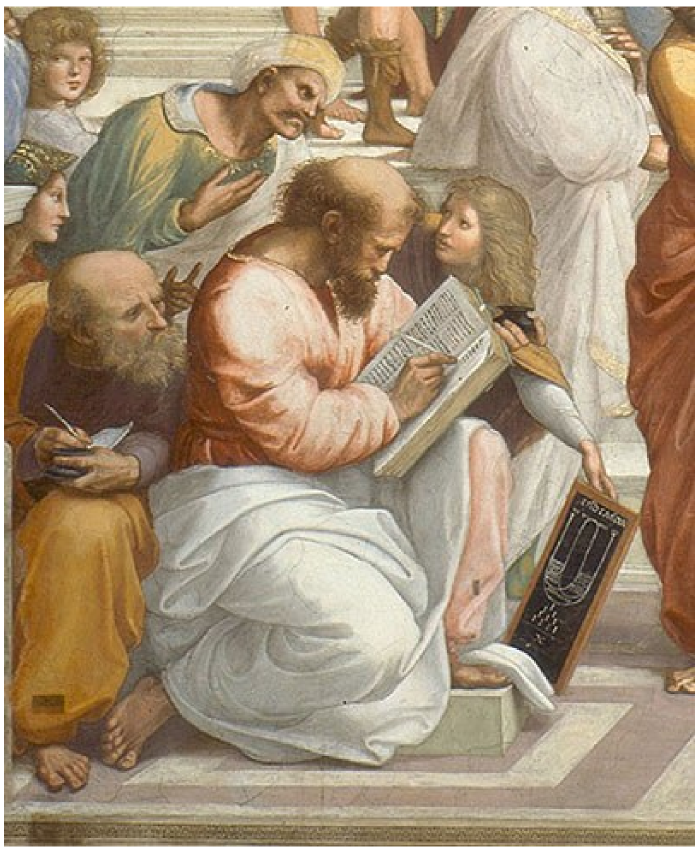


Figure 138: In the lower left corner of the fresco *The School of Athens* by Raphael, the figure in white writing in the book represents Pythagoras. Alongside him, to the right, the figure with the long, light-brown hair is said to depict Archimedes.

Learning Objectives

After completing this section, you should be able to:

1. Apply the Pythagorean Theorem to find the missing sides of a right triangle.

2. Apply the 30° - 60° - 90° and 45° - 45° - 90° right triangle relationships to find the missing sides of a triangle.
3. Apply trigonometric ratios to find missing parts of a right triangle.
4. Solve application problems involving trigonometric ratios.

This is another excerpt from Raphael's *The School of Athens*. The man writing in the book represents Pythagoras, the namesake of one of the most widely used formulas in geometry, engineering, architecture, and many other fields, the Pythagorean Theorem. However, there is evidence that the theorem was known as early as 1900–1100 BC by the Babylonians. The Pythagorean Theorem is a formula used for finding the lengths of the sides of right triangles.

Born in Greece, Pythagoras lived from 569–500 BC. He initiated a cult-like group called the Pythagoreans, which was a secret society composed of mathematicians, philosophers, and musicians. Pythagoras believed that everything in the world could be explained through numbers. Besides the Pythagorean Theorem, Pythagoras and his followers are credited with the discovery of irrational numbers, the musical scale, the relationship between music and mathematics, and many other concepts that left an immeasurable influence on future mathematicians and scientists.

The focus of this section is on right triangles. We will look at how the **Pythagorean Theorem** is used to find the unknown sides of a right triangle, and we will also study the *special* triangles, those with set ratios between the lengths of sides. By ratios we mean the relationship of one side to another side. When you think about ratios, you should think about fractions. A fraction is a ratio, the ratio of the numerator to the denominator. Finally, we will preview trigonometry. We will learn about the basic trigonometric functions, sine, cosine and tangent, and how they are used to find not only unknown sides but unknown angles, as well, with little information.

Pythagorean Theorem

The Pythagorean Theorem is used to find unknown sides of right triangles. The theorem states that the sum of the squares of the two legs of a right triangle equals the square of the hypotenuse (the longest side of the right triangle).

FORMULA

The Pythagorean Theorem states

$$a^2 + b^2 = c^2$$

where a and b are two sides (legs) of a right triangle and c is the hypotenuse, as shown.

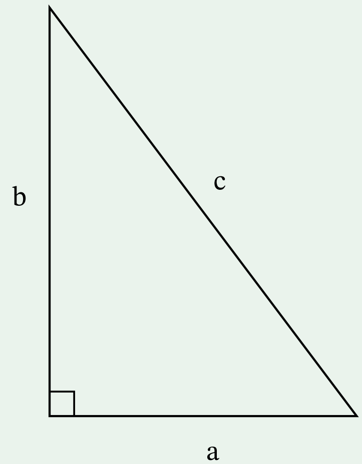


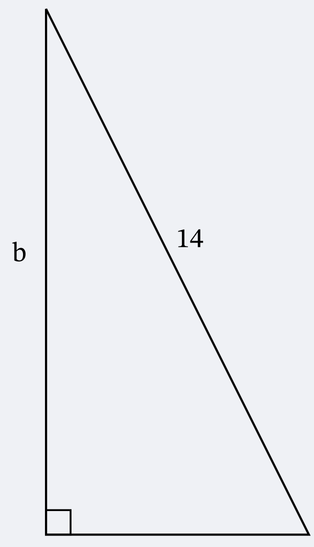
Figure 139: Pythagorean Right Triangle

For example, given that side $a = 6$, and side $b = 8$, we can find the measure of side c using the Pythagorean Theorem. Thus,

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
(6)^2 + (8)^2 &= c^2 \\
36 + 64 &= c^2 \\
100 &= c^2 \\
\sqrt{100} &= \sqrt{c^2} \\
10 &= c
\end{aligned}
$$

Example 1 **Using the Pythagorean Theorem**

Find the length of the missing side of the triangle .



Solution

Using the Pythagorean Theorem, we have

$$\begin{aligned} (6)^2 + b^2 &= (14)^2 \\ 36 + b^2 &= 196 \\ b^2 &= 196 - 36 \\ b^2 &= 160 \\ b &= \pm\sqrt{160} \\ &= 4\sqrt{10} = 12.65 \end{aligned}$$

When we take the square root of a number, the answer is usually both the positive and negative root. However, lengths cannot be negative, which is why we only consider the positive root.

Distance

The applications of the Pythagorean Theorem are countless, but one especially useful application is that of distance. In fact, the distance formula stems directly from the theorem. It works like this:

In , the problem is to find the distance between the points $(-3, -1)$ and $(3, 2)$. We call the length from point $(-3, -1)$ to point $(3, -1)$ side a, and the length from point $(3, -1)$ to point $(3, 2)$ side b. To find side c, we use the distance formula and we will explain it relative to the Pythagorean Theorem. The distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, such that $(x_2 - x_1)$ is a substitute for a in the Pythagorean Theorem and is equal to $3 - (-3) = 6$; and $(y_2 - y_1)$ is a substitute for b in the Pythagorean Theorem and is equal to $2 - (-1) = 3$. When we plug in these numbers to the distance formula, we have

$$\begin{aligned} d &= \sqrt{(3-(-3))^2 + (2-(-1))^2} \\ &= \sqrt{(6)^2 + (3)^2} = \sqrt{36+9} \\ &= \sqrt{45} = 3\sqrt{5} = 6.7 \end{aligned}$$

Thus, $d = c$, the hypotenuse, in the Pythagorean Theorem.

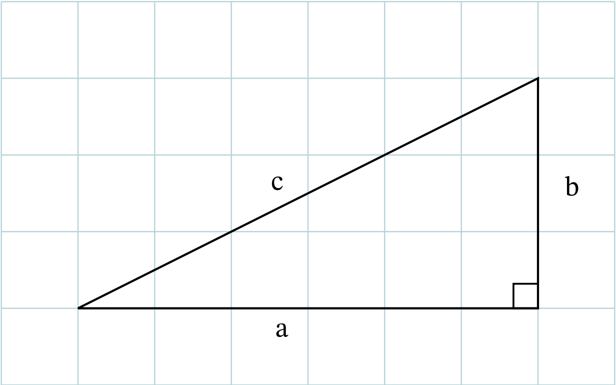
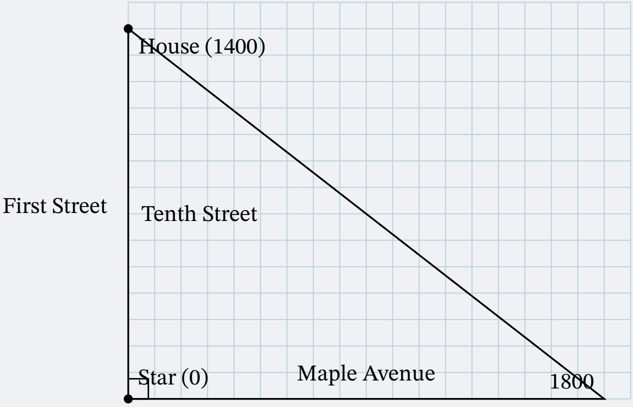


Figure 141: Distance

Example 2 **Calculating Distance Using the Distance Formula**

You live on the corner of First Street and Maple Avenue, and work at Star Enterprises on Tenth Street and Elm Drive . You want to calculate how far you walk to work every day and how it compares to the actual distance (as the crow flies). Each block measures 200 ft by 200 ft.

Right Triangle on Grid - Elm Drive and First Street



Solution

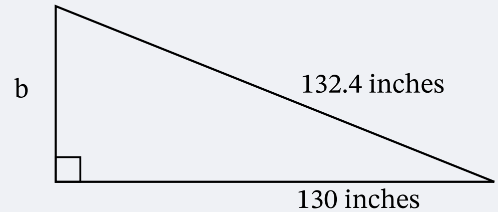
You travel 7 blocks south and 9 blocks west. If each block measures 200 ft by 200 ft, then $9(200) + 7(200) = 1,800 \text{ ft} + 1,400 \text{ ft} = 3,200 \text{ ft}$.

As the crow flies, use the distance formula. We have

$$\begin{aligned} d &= \sqrt{(1,800-0)^2+(1,400-0)^2} \\ &= \sqrt{3,240,000+1,960,000} \\ &= \sqrt{5,200,000} \\ &= 2280.4 \text{ ft} \end{aligned}$$

Example 3 **Calculating Distance with the Pythagorean Theorem**

The city has specific building codes for wheelchair ramps. Every vertical rise of 1 in requires that the horizontal length be 12 inches. You are constructing a ramp at your business. The plan is to make the ramp 130 inches in horizontal length and the slanted distance will measure approximately 132.4 inches . What should the vertical height be?



Solution

The Pythagorean Theorem states that the horizontal length of the base of the ramp, side a, is 130 in. The length of c, or the length of the hypotenuse, is 132.4 in. The length of the height of the triangle is side b.

Then, by the Pythagorean Theorem, we have:

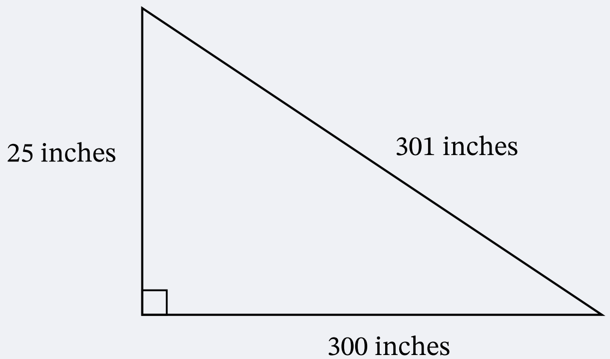
$$\begin{aligned} a^2+b^2 &= c^2 \\ (130)^2+b^2 &= (132.4)^2 \\ 16,900+b^2 &= 17,529.76 \\ b^2 &= 17,529.76-16,900 \\ b^2 &= 629.8 \\ b &= \sqrt{629.8} = 25 \end{aligned}$$

If you construct the ramp with a 25 in vertical rise, will it fulfill the building code? If not, what will have to change?

The building code states 12 in of horizontal length for each 1 in of vertical rise. The vertical rise is 25 in, which means that the horizontal length has to be $12(25) = 300$ in . So, no, this will not pass the code. If you must keep the vertical rise at 25 in, what will the other dimensions have to be? Since we need a minimum of 300 in for the horizontal length:

$$\begin{aligned} (300)^2 + (25)^2 = & \quad c^2 \\ 90,625 = & \quad c^2 \\ \sqrt{90{,}625} = & \ c = 301 \text{ in} \end{aligned}$$

The new ramp will look like .



30° - 60° - 90° **Triangles**

In geometry, as in all fields of mathematics, there are always special rules for special circumstances. An example is the *perfect square* rule in algebra. When expanding an expression like $(2x + 5y)^2$, we do not have to expand it the long way:

$$\begin{aligned} (2x + 5y)^2 &= (2x + 5y)(2x + 5y) \\ &= (2x)^2 + 10xy + 10xy + (5y)^2 \\ &= 4x^2 + 20xy + 25y^2 \end{aligned}$$

If we know the perfect square formula, given as

$$(a + b)^2 = a^2 + 2ab + b^2,$$

we can skip the middle step and just start writing down the answer. This may seem trivial with problems like $(a + b)^2$. However, what if you have a problem like $\left(2\sqrt{3} + 3\sqrt[3]{31.8c}\right)^2$? That is a different story. Nevertheless, we use the same perfect square formula. The same idea applies in geometry. There are special formulas and procedures to apply in certain types of problems. What is needed is to remember the formula and remember the kind of problems that fit. Sometimes we believe that because a formula is labeled *special,* we will rarely have use for it. That assumption is incorrect. So, let us identify the 30° - 60° - 90° triangle and find out why it is special.

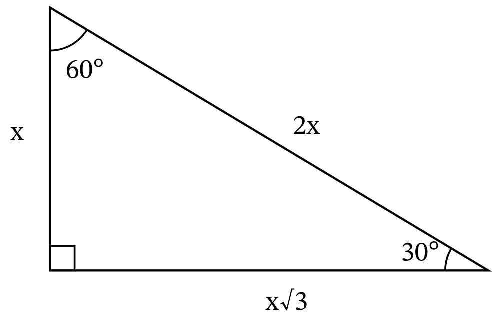
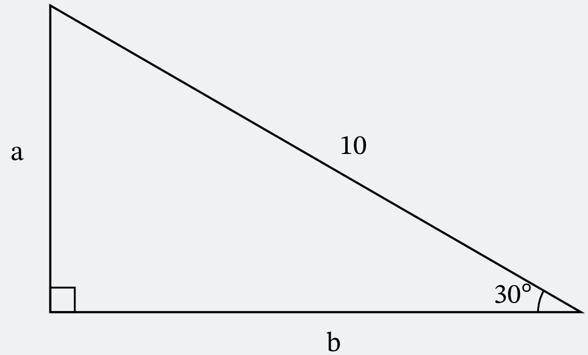


Figure 145: The 30° - 60° - 90°

We see that the shortest side is opposite the smallest angle, and the longest side, the hypotenuse, will always be opposite the right angle. There is a set ratio of one side to another side for the 30° - 60° - 90° triangle given as $1 : \sqrt{3} : 2$, or $x : x\sqrt{3} : 2x$. Thus, you only need to know the length of one side to find the other two sides in a 30° - 60° - 90° triangle.

Example 4 **Finding Missing Lengths in a 30° - 60° - 90° Triangle**

Find the measures of the missing lengths of the triangle .



Solution

We can see that this is a 30° - 60° - 90° triangle because we have a right angle and a 30° angle. The remaining angle, therefore, must equal 60°. Because this is a special triangle, we have the ratios of the sides to help us identify the missing lengths. Side a is the shortest side, as it is opposite the smallest angle 30°, and we can substitute $a = x$. The ratios are $x : x\sqrt{3} : 2x$. We have the hypotenuse equaling 10, which corresponds to side c, and side c is equal to $2x$. Now, we must solve for x:

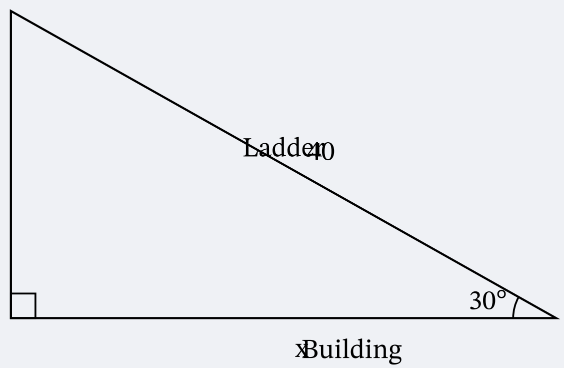
$$\begin{aligned} 2x &= 10 \\ x &= \ \ 5 \end{aligned}$$

Side b is equal to $x\sqrt{3}$ or $5\sqrt{3}$. The lengths are $5, 5\sqrt{3}, 10$.

Example 5 **Applying 30° - 60° - 90° Triangle to the Real World**

A city worker leans a 40-foot ladder up against a building at a 30° angle to the ground . How far up the building does the ladder reach?

Ladder Against Building



Solution

We have a 30° - 60° - 90° triangle, and the hypotenuse is 40 ft. This length is equal to $2x$, where x is the shortest side. If $2x = 40$, then $x = 20$. The ladder is leaning on the wall 20 ft up from the ground.

45° - 45° - 90° **Triangles**

The 45° - 45° - 90° triangle is another *special* triangle such that with the measure of one side we can find the measures of all the sides. The two angles adjacent to the 90° angle are equal, and each measures 45°. If two angles are equal, so are their opposite sides. The ratio among sides is $1 : 1 : \sqrt{2}$, or $x : x : x\sqrt{2}$, as shown.

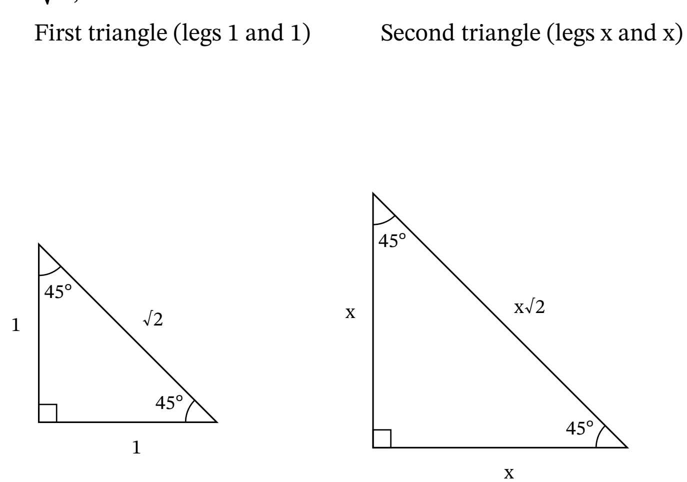
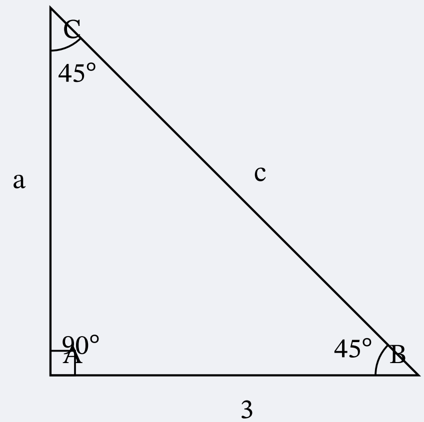


Figure 148: 45° - 45° - 90° Triangles

Example 6 **Finding Missing Lengths of a** $45°$ - $45°$ - $90°$ **Triangle**

Find the measures of the unknown sides in the triangle .

Right Triangle



Solution

Because we have a $45°$ - $45°$ - $90°$ triangle, we know that the two legs are equal in length and the hypotenuse is a product of one of the legs and $\sqrt{2}$. One leg measures 3, so the other leg, a, measures 3. Remember the ratio of $x : x : x\sqrt{2}$. Then, the hypotenuse, c, equals $3\sqrt{2}$.

Trigonometry Functions

Trigonometry developed around 200 BC from a need to determine distances and to calculate the measures of angles in the fields of astronomy and surveying. Trigonometry is about the relationships (or ratios) of angle measurements to side lengths in primarily right triangles. However, trigonometry is useful in calculating missing side lengths and angles in other triangles and many applications.

CHECKPOINT

NOTE: You will need either a scientific calculator or a graphing calculator for this section. It must have the capability to calculate trigonometric functions and express angles in degrees.

Trigonometry is based on three functions. We title these functions using the following abbreviations:

- sin = sine
- cos = cosine
- tan = tangent

Letting $r = \sqrt{x^2 + y^2}$, which is the hypotenuse of a right triangle, we have . The functions are given in terms of x, y, and r, and in terms of sides relative to the angle, like opposite, adjacent, and the hypotenuse.

$\sin\theta = \frac{y}{r} = \frac{opp}{hyp}$	$\cos\theta = \frac{x}{r} = \frac{adj}{hyp}$	$\tan\theta = \frac{y}{x} = \frac{opp}{adj}$

We will be applying the sine function, cosine function, and tangent function to find side lengths and angle measurements for triangles we cannot solve using any of the techniques we have studied to this point. In , we have an illustration mainly to identify r and the sides labeled x and y.

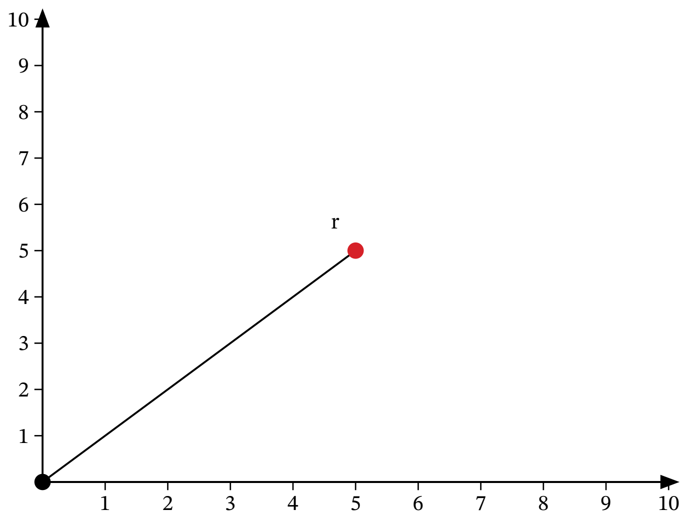


Figure 150: Angle θ

An angle θ sweeps out in a counterclockwise direction from the positive x-axis and stops when the angle reaches the desired measurement. That ray extending from the origin that marks $\theta°$ is called the terminal side because that is where the angle terminates. Regardless of the information given in the triangle, we can find all missing sides and angles using the trigonometric functions. For example, in , we will solve for the missing sides.

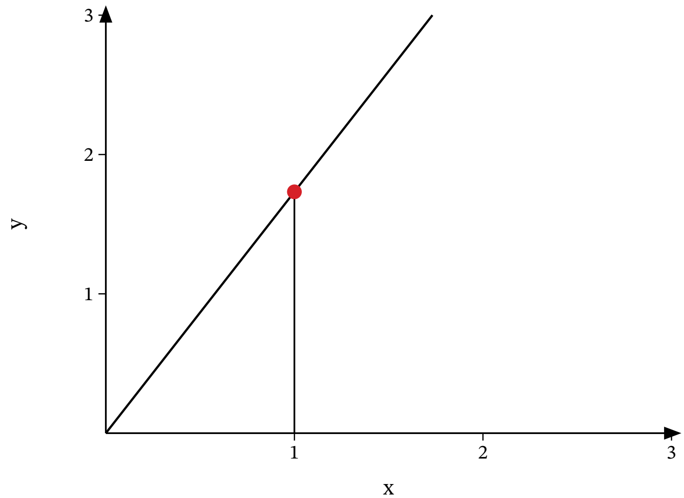


Figure 151: Solving for Missing Sides

Let's use the trigonometric functions to find the sides x and y. As long as your calculator mode is set to degrees, you do not have to enter the degree symbol. First, let's solve for y.

We have $\sin\theta = \frac{y}{r}$, and $\theta = 60°$. Then,

$$\begin{aligned} \sin 60° &= \frac{y}{2} \\ 2\sin 60° &= y \\ 1.732 &= y \\ \sqrt{3} &= y \end{aligned}$$

Next, let's find x. This is the cosine function. We have $\cos\theta = \frac{x}{r}$. Then,

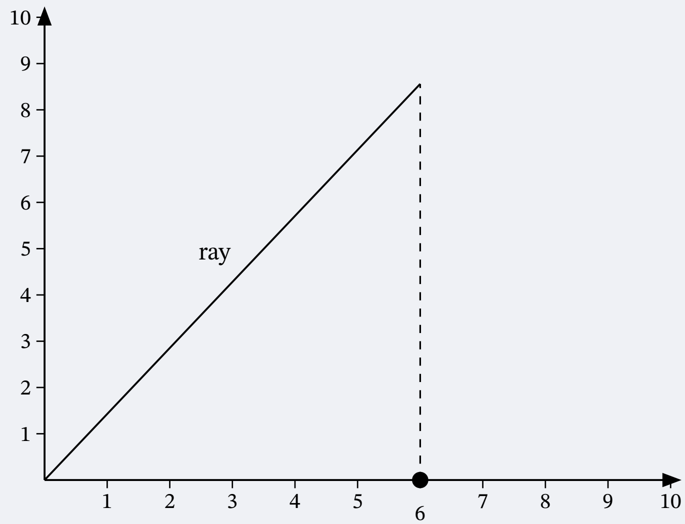
$$\begin{aligned} \cos 60° &= \frac{x}{2} \\ 2\cos 60° &= x \\ &= 1 \end{aligned}$$

Now we have all sides, $1, \sqrt{3}, 2$. You can also check the sides using the 30° - 60° - 90° ratio of $1 : \sqrt{3} : 2$. is a list of common angles, which you should find helpful.

$\sin 0° = 0$	$\cos 0° = 1$
$\sin 30° = \frac{1}{2}$	$\cos 30° = \frac{\sqrt{3}}{2}$
$\sin 45° = \frac{\sqrt{2}}{2}$	$\cos 45° = \frac{\sqrt{2}}{2}$
$\sin 60° = \frac{\sqrt{3}}{2}$	$\cos 60° = \frac{1}{2}$
$\sin 90° = 1$	$\cos 90° = 0$

Example 7 **Using Trigonometric Functions**

Find the lengths of the missing sides for the triangle .



Solution

We have a 55° angle, and the length of the triangle on the x-axis is 6 units.

Step 1: To find the length of r, we can use the cosine function, as $\cos\theta = \frac{x}{r}$. We manipulate this equation a bit to solve for r:

$$\begin{aligned} \cos(55°) &= \frac{6}{r} \\ r\cos(55°) &= 6 \\ r &= \frac{6}{\cos(55°)} \\ r &= \frac{6}{0.5736} = 10.46 \end{aligned}$$

Step 2: We can use the Pythagorean Theorem to find the length of y. Prove that your answers are correct by using other trigonometric ratios:

$$\begin{aligned} 6^2 + y^2 &= 10.46^2 \\ y^2 &= 109.4 - 36 \\ y &= 8.57 \end{aligned}$$

Step 3: Now that we have y, we can use the sine function to prove that r is correct. We have $\sin\theta = \frac{y}{r}$.

$$\begin{aligned}
\sin(55°) &= \frac{8.57}{r} \\
r\sin(55°) &= 8.57 \\
r &= \frac{8.57}{\sin(55°)} \\
&= \frac{8.57}{0.819} = 10.46
\end{aligned}$$

To find angle measurements when we have two side measurements, we use the inverse trigonometric functions symbolized as $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$. The –1 looks like an exponent, but it means inverse. For example, in the previous example, we had $x = 6$ and $r = 10.46$. To find what angle has these values, enter the values for the inverse cosine function $\cos^{-1}\left(\frac{x}{r}\right)$ in your calculator:

$$\cos^{-1}\left(\frac{6}{10.46}\right) = 55°.$$

You can also use the inverse sine function and enter the values of $\sin^{-1}\left(\frac{y}{r}\right)$ in your calculator given $y = 8.57$ and $r = 10.46$. We have

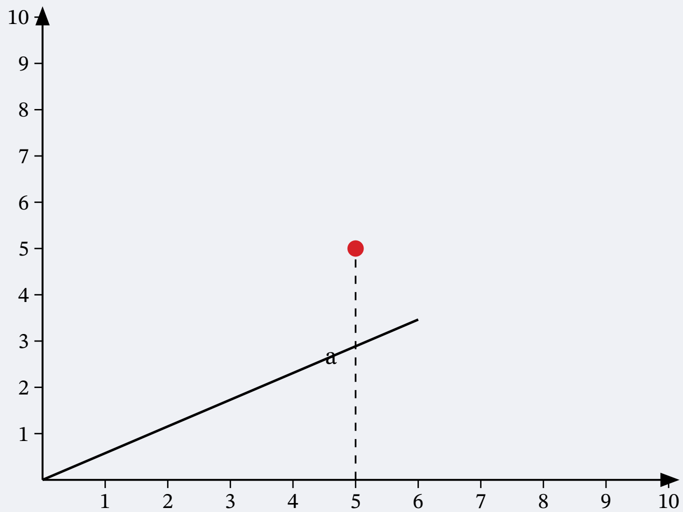
$$\sin^{-1}\left(\frac{8.57}{10.46}\right) = 55°.$$

Finally, we can also use the inverse tangent function. Recall $\tan\theta = \frac{y}{x}$. We have

$$\tan^{-1}\left(\frac{8.57}{6}\right) = 55°.$$

Example 8 **Solving for Lengths in a Right Triangle**

Solve for the lengths of a right triangle in which $\theta = 30°$ and $r = 6$.



Solution

Step 1: To find side a, we use the sine function:

$$\sin 30° = \frac{a}{6}$$

$$6 \sin 30° = a = 3$$

Step 2: To find b, we use the cosine function:

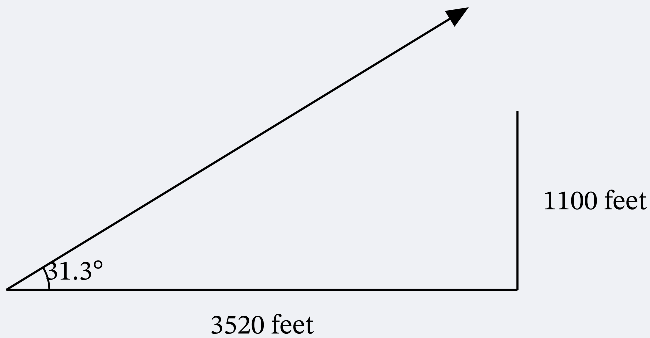
$$\cos 30° = \frac{b}{6}$$

$$6 \cos 30° = b = 5.196$$

Step 3: Since this is a 30° - 60° - 90° triangle and side b should equal $x\sqrt{3}$, if we input 3 for x, we have $b = 3\sqrt{3}$. Put this in your calculator and you will get $3\sqrt{3} = 5.196$.

Example 9 **Finding Altitude**

A small plane takes off from an airport at an angle of 31.3° to the ground. About two-thirds of a mile (3,520 ft) from the airport is an 1,100-ft peak in the flight path of the plane . If the plane continues that angle of ascent, find its altitude when it is above the peak, and how far it will be above the peak.



Solution

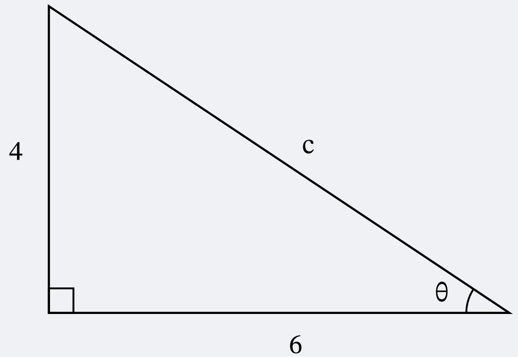
To solve this problem, we use the tangent function:

$$\tan 31.3^\circ = \frac{x}{3{,}520}$$
$$3,520 \tan 31.3^\circ = 2,140$$

The plane's altitude when passing over the peak is 2,140 ft, and it is 1,040 ft above the peak.

Example 10 **Finding Unknown Sides and Angles**

Suppose you have two known sides, but do not know the measure of any angles except for the right angle . Find the measure of the unknown angles and the third side.



Solution

Step 1: We can find the third side using the Pythagorean Theorem:

$$6^2 + 4^2 = c^2$$
$$52 = c^2$$
$$2\sqrt{13} = c$$

Now, we have all three sides.

Step 2: To find θ, we will first find $\sin\theta$.

$$\sin\theta = \frac{opp}{hyp} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}.$$

The angle θ is the angle whose sine is $\frac{2}{\sqrt{13}}$.

Step 3: To find θ, we use the inverse sine function:

$$\theta = \sin^{-1}\left(\frac{2}{\sqrt{13}}\right) = 33.7^\circ$$

Step 4: To find the last angle, we just subtract: $180^\circ - 90^\circ - 33.7^\circ = 56.3^\circ$.

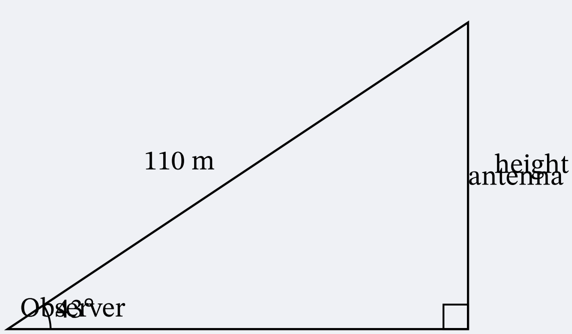
Angle of Elevation and Angle of Depression

Other problems that involve trigonometric functions include calculating the **angle of elevation** and the **angle of depression**. These are very common applications in everyday life. The angle of elevation is the angle formed by a horizontal line and the line of sight from an observer to some object at a higher level. The angle of depression is the angle formed by a horizontal line and the line of sight from an observer to an object at a lower level.

Example 11 **Finding the Angle of Elevation**

A guy wire of length 110 meters runs from the top of an antenna to the ground . If the angle of elevation of an observer to the top of the antenna is 43°, how high is the antenna?

Right Triangle - Antenna and Guy Wire



Solution

We are looking for the height of the tower. This corresponds to the y-value, so we will use the sine function:

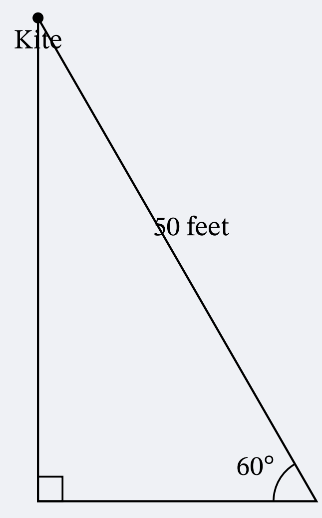
$$\begin{aligned} \sin 43° &= \frac{y}{110} \\ 110 \sin 43° &= y \\ 75 &= y \end{aligned}$$

The tower is 75 m high.

Example 12 **Finding Angle of Elevation**

You are sitting on the grass flying a kite on a 50-foot string . The angle of elevation is 60°. How high above the ground is the kite?

Right Triangle with Kite



Solution

We can solve this using the sine function, $\sin\theta = \frac{opp}{hyp}$.

$$\begin{aligned} \sin 60° &= \frac{x}{50} \\ 50 \sin 60° &= x \\ &= 43.3 \text{ ft} \end{aligned}$$

PEOPLE IN MATHEMATICS

Pythagoras and the Pythagoreans

The Pythagorean Theorem is so widely used that most people assume that Pythagoras (570–490 BC) discovered it. The philosopher and mathematician uncovered evidence of the right triangle concepts in the teachings of the Babylonians dating around 1900 BC. However, it was Pythagoras who found countless applications of the theorem leading to advances in geometry, architecture, astronomy, and engineering.

Among his accolades, Pythagoras founded a school for the study of mathematics and music. Students were called the Pythagoreans, and the school's teachings could be classified as a religious indoctrination just as much as an academic experience. Pythagoras believed that spirituality and science coexist, that the intellectual mind is superior to the senses, and that intuition should be honored over observation.

Pythagoras was convinced that the universe could be defined by numbers, and that the natural world was based on mathematics. His primary belief was *All is Number.* He even attributed certain qualities to certain numbers, such as the number 8 represented justice and the number 7 represented wisdom. There was a quasi-mythology that surrounded Pythagoras. His followers thought that he was more of a spiritual being, a sort of mystic that was all-knowing and could travel through time and space. Some believed that Pythagoras had mystical powers, although these beliefs were never substantiated.

Pythagoras and his followers contributed more ideas to the field of mathematics, music, and astronomy besides the Pythagorean Theorem. The Pythagoreans are credited with the discovery of irrational numbers and of proving that the morning star was the planet Venus and not a star at all. They are also credited with the discovery of the musical scale and that different strings made different sounds based on their length. Some other concepts attributed to the Pythagoreans include the properties relating to triangles other than the right triangle, one of which is that the sum of the interior angles of a triangle equals 180°. These geometric principles, proposed by the Pythagoreans, were proven 200 years later by Euclid.

WHO KNEW?

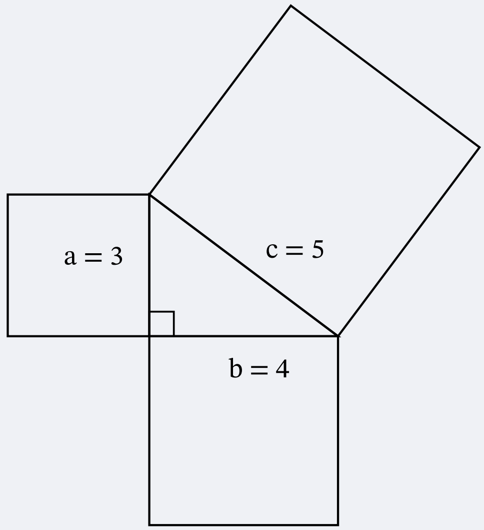
A Visualization of the Pythagorean Theorem

In , which is one of the more popular visualizations of the Pythagorean Theorem, we see that square a is attached to side a square b is attached to side b and the largest square, square c, is attached to side c. Side a measures 3 cm in length, side b measures 4 cm in length, and side c measures 5 cm in length. By definition, the area of square a measures 9 square units, the area of square b measures 16 square units, and the area of square c measures 25 square units. Substitute the values given for the areas of the three squares into the Pythagorean Theorem and we have

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \end{aligned}$$

Thus, the sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse, as stated in the Pythagorean Theorem.

Right Triangle with Squares on Each Side



Key Terms

- right triangle
- sine
- cosine
- tangent

Key Concepts

- The Pythagorean Theorem is applied to right triangles and is used to find the measure of the legs and the hypotenuse according the formula $a^2 + b^2 = c^2$, where c is the hypotenuse.
- To find the measure of the sides of a *special* angle, such as a 30° - 60° - 90° triangle, use the ratio $x : x\sqrt{3} : 2x$, where each of the three sides is associated with the opposite angle and $2x$ is associated with the hypotenuse, opposite the 90° angle.
- To find the measure of the sides of the second *special* triangle, the 45° - 45° - 90° triangle, use the ratio $x : x : x\sqrt{2}$, where each of the three sides is associated with the opposite angle and $x\sqrt{2}$ is associated with the hypotenuse, opposite the 90° angle.
- The primary trigonometric functions are $\sin\theta = \frac{opp}{hyp}$, $\cos\theta = \frac{adj}{hyp}$, and $\tan\theta = \frac{opp}{adj}$.

- Trigonometric functions can be used to find either the length of a side or the measure of an angle in a right triangle, and in applications such as the angle of elevation or the angle of depression formed using right triangles.

Formula

The Pythagorean Theorem states

$$a^2 + b^2 = c^2$$

where a and b are two sides (legs) of a right triangle and c is the hypotenuse.

Projects

1. One of the reasons so many formulas in geometry were discovered was because of the importance in finding measurements of lengths, areas, perimeter, and angles. Find at least five examples of how geometry can be used in practical applications today.
2. Who were the Pythagoreans? Why did this society exist? Explore what they did and discuss some of their beliefs.