Chapter 5: Algebra

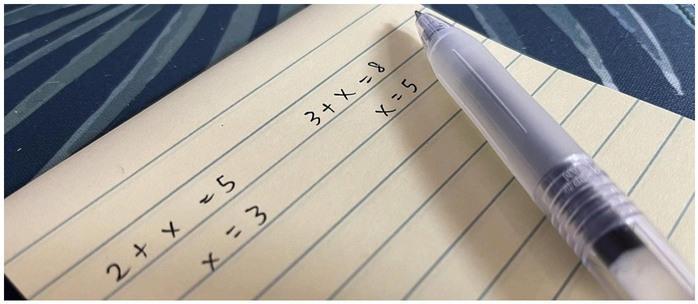


Figure 1: In these algebraic equations, the x represents different numbers.

The jump from arithmetic to algebra can be a difficult one for many students. Many students struggle with the idea that mathematics can include situations that aren't static and do change. In elementary arithmetic, a situation such as: $5 + 3 = \_\_\_\_$

is a static situation and will yield the answer of 8 every time. However, a situation such as: $5x + 3 = \_\_\_\_$

can yield many different answers because the answer depends on what amount (number) that x represents. Since the value of x can vary (represent different values), it is known as a variable.

Algebra is useful to better model real life situations. In the first equation shown, $5 + 3 = \_\_\_\_$ can only model situations where you add those two numbers together. For example, if your uncle gives you five dollars and your aunt gives you three dollars, then you will always receive eight dollars. The second equation $5x + 3 = \_\_\_\_$ can model more complex situations. For example, you wish to buy a game that costs $38 but you only have three dollars. Your uncle will pay you five dollars an hour to work for him. If you've worked five hours, have you earned enough money? If not, how many hours will you have to work?

Algebra and algebraic thinking open up a world of possibilities that arithmetic alone cannot do.

5.1 Algebraic Expressions

Figure 2: Two college graduates!

Learning Objectives

After completing this section, you should be able to:

1. Convert between written and symbolic algebraic expressions and equations.
2. Simplify and evaluate algebraic expressions.
3. Add and subtract algebraic expressions.
4. Multiply and divide algebraic expressions.

Algebraic expressions are the building blocks of algebra. While a numerical expression (also known as an arithmetic expression) like $5 + 3$ can represent only a single number, an algebraic expression such as $5x + 3$ can represent many different numbers. This section will introduce you to algebraic expressions, how to create them, simplify them, and perform arithmetic operations on them.

Algebraic Expressions and Equations

Xavier and Yasenia have the same birthday, but they were born in different years. This year Xavier is 20 years old and Yasenia is 23, so Yasenia is three years older than Xavier. When Xavier was 15, Yasenia was 18. When Xavier will be 33, Yasenia will be 36. No matter what Xavier's age is, Yasenia's age will always be 3 years more.

In the language of algebra, we say that Xavier's age and Yasenia's age are variable and the 3 is a constant. The ages change, or vary, so age is a **variable**. The 3 years between them always stays the same or has the same value, so the age difference is the **constant**. In algebra, letters of the alphabet are used to represent variables. The letters most often used for variables are x, y, z, a, b, and c. Suppose we call Xavier's age x. Then we could use $x + 3$ to represent Yasenia's age, as shown in the table below.

Xavier's Age	Yasenia's Age
15	18
20	23
33	36
x	$x + 3$

To write algebraically, we need some symbols as well as numbers and variables. The symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division are summarized in , along with words we use for the operations and the result.

Operation	Notation	Say:	The result is…
Addition	$a + b$	a plus b	The sum of a and b
Subtraction	$a - b$	a minus b	The difference of a and b
Multiplication	$a \cdot b$, $(a)(b)$, $(a)b$, $a\,(b)$, ab, ba	a times b	The product of a and b
Division	$a \div b$, a/b	a divided by b	The quotient of a and b

CHECKPOINT

In algebra, the cross symbol (x) *is normally not used to show multiplication because that symbol could cause confusion. For example, does* $3xy$ *mean* $3 \times y$ *(three times y) or* $3 \cdot x \cdot y$ *(three times x times y)? To make it clear, use* $\cdot$ *or parentheses for multiplication.*

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum **of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5 + 3$.
- The *difference **of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9 - 2$.
- The *product **of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4 \cdot 8$.
- The *quotient **of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20 \div 5$.

Example 1 **Translating from Algebra to Words**

Translate the following algebraic expressions from algebra into words.

1. $12 + 14$
2. $(30)(5)$
3. $64 \div 8$
4. $x - y$

Solution

1. This could be translated as 12 plus 14 OR the sum of 12 and 14.
2. This could be translated as 30 times 5 OR the product of 30 and 5.
3. This could be translated as 64 divided by 8 OR the quotient of 64 and 8.
4. This could be translated as x minus y OR the difference of x and y.

Example 2 **Translating from Words to Algebra**

Translate the following phrases from words into algebraic expressions.

1. The difference of 47 and 19
2. 72 divided by 9
3. The sum of m and n
4. 13 times 7

Solution

1. These words could be translated as $47 - 19$.
2. These words could be translated as $72 \div 9$.
3. These words could be translated as $m + n$.
4. These words could be translated as $(13)(7)$.

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. "Running very fast" is a phrase, but "The football player was running very fast" is a sentence. A sentence has a subject and a verb. In algebra, we have **expressions** and equations. Example 1 and Example 2 used expressions. An expression is like an English phrase. Notice that the English phrases do not form a complete sentence because the phrase does not have a verb. The following table has examples of expressions, which are numbers, variables, or combinations of numbers and variables using operation symbols.

Expression	Words	English Phrase
$3 + 5$	3 plus 5	The sum of three and five
$n - 1$	n minus one	The difference of n and one
$6 \cdot 7$	6 times 7	The product of six and seven
$x \div y$	x divided by y	The quotient of x and y

Example 3 **Translating from an English Phrase to an Expression**

Translate the following phrases from words into algebraic expressions.

1. Seven more than a number n.
2. A number n times itself.
3. Six times a number n, plus two more.
4. The cost of postage is a flat rate of 10 cents for every parcel, plus 34 cents per ounce x.

Solution

1. $n + 7$
2. $n \cdot n$ or n^2
3. $6n + 2$
4. $10 + 34x$

An **equation** is two expressions linked with an **equal sign** (the symbol $=$). When two quantities have the same value, we say they are equal and connect them with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. So, $a = b$ is read "a is equal to b." The following table has some examples of equations.

Equation	English Sentence
$3 + 5 = 8$	The sum of three and five is equal to eight.
$n - 1 = 14$	n minus one equals fourteen.
$6 \cdot 7 = 42$	The product of six and seven is equal to forty-two.
$x = 53$	x is equal to fifty-three.
$y + 9 = 2y - 3$	y plus nine is equal to two times y minus three.

Example 4 **Translating from an English Sentence to an Equation**

Translate the following sentences from words into algebraic equations.

1. Two times x is 6.
2. n plus 2 is equal to n times 3.
3. The quotient of 35 and 7 is 5.
4. Sixty-seven minus x is 56.

Solution

1. $2x = 6$
2. $n + 2 = 3n$
3. $35 \div 7 = 5$
4. $67 - x = 56$

WHO KNEW?

The Use of Variables

French philosopher and mathematician René Descartes (1596–1650) is usually given credit for the use of the letters x, y, and z to represent unknown quantities in algebra. He introduced these ideas in his publication of *La Geometrie*, which was printed in 1637. In this publication, he also used the letters a, b, and c to represent known quantities. There is a (possibly fictitious) story that, when the book was being printed for the first time, the printer began to run short of the last three letters of the alphabet. So the printer asked Descartes if it mattered which of x, y, or z were used for the mathematical equations in the book. Descartes decided it made no difference to him; so the printer decided to use x predominantly for the mathematics in the book, because the letters y and z would occur more often in the body of the text (written in French) than the letter x would! This might explain why the letter x is still used today as the most common variable to represent unknown quantities in algebra.

Simplifying and Evaluating Algebraic Expressions

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \cdot 2 + 1$ we would first multiply $4 \cdot 2$ to get 8 and then add 1 to get 9. We have introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. Consider $2 + 7 \cdot 3$. Do you add first or multiply first? Do you get different answers?

Add first: $9 \cdot 3 = 27$	Multiply first: $2 + 21 = 23$	Which one is correct?

Early on, mathematicians realized the need to establish some guidelines when performing arithmetic operations to ensure that everyone would get the same answer. Those guidelines are called the order of operations and are listed in the table below.

Step 1: Parentheses and Other Grouping Symbols	Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
Step 2: Exponents	Simplify all expressions with exponents.
Step 3: Multiplication and Division	Perform all multiplication and division in order from left to right. These operations have equal priority.
Step 4: Addition and Subtraction	Perform all addition and subtraction in order from left to right. These operations have equal priority.

CHECKPOINT

You may have heard about Please Excuse My Dear Aunt Sally or PEMDAS. Be careful to notice *in Steps 3 and 4 in the table above that multiplication and division, as well as addition and subtraction, happen in order from LEFT to RIGHT. It is possible, for example, to have PEDMAS or PEMDSA. The PEMDAS trick can be misleading if not fully understood!*

Example 5 **Making a Numerical Equation True Using the Order of Operations**

Use parentheses to make the following statements true.

1. $17 - 10 + 3 = 10$
2. $2 \cdot 26 - 7 = 38$
3. $8 + 12 \div 5 - 3 = 14$
4. $5 + 2^3 \cdot 7 = 91$

Solution

1. Add the parentheses around the $17 - 10$. Then you have $(17 - 10) + 3 = 7 + 3 = 10$.
2. Add the parentheses around the $26 - 7$. Then you have $2 \cdot (26 - 7) = 2 \cdot 19 = 38$.
3. Add the parentheses around the $5 - 3$. Then you have $8 + 12 \div (5 - 3) = 8 + 12 \div 2 = 8 + 6 = 14$.
4. Add the parentheses around the $5 + 2^3$. Then you have $\left(5 + 2^3\right) \cdot 7 = (5 + 8) \cdot 7 = 13 \cdot 7 = 91$.

In the last example, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

Example 6 **Evaluating and Simplifying an Expression**

1. Evaluate $3x + 5$ when $x = 2$.
2. Evaluate $x^2 + 3x + 1$ when $x = 2$.

Solution

1. To evaluate, let $x = 2$ in the expression, and then simplify: $3(2) + 5 = 6 + 5 = 11$.
2. To evaluate, let $x = 2$ in the expression, and then simplify: $2^2 + 3(2) + 1 = 4 + 6 + 1 = 11$.

Operations of Algebraic Expressions

Algebraic expressions are made up of **terms**. A term is a constant or the product of a constant and one or more variables. Examples of terms are 7, y, $5x^2$, $9a$, and b^5. The constant that multiplies the variable is called the **coefficient**. Think of the coefficient as the number in front of the variable. Consider the algebraic expressions $5x^2$, which has a coefficient of 5, and $9a$, which has a coefficient of 9. If there is no number listed in front of the variable, then the coefficient is 1 since $x = 1 \cdot x$.

Some terms share common traits. When two terms are constants or have the same variable and exponent, we say they are **like terms**. If there are like terms in an expression, you can simplify the expression by combining the like terms. We add the coefficients and keep the same variable.

Example 7 **Adding Algebraic Expressions**

Add $\left(x^2 + 4x - 9\right) + \left(3x^2 - x + 12\right)$.

Solution

Step 1: Add the terms in any order and get the same result (think: $2 + 3 = 3 + 2$) and drop the parentheses:

$$x^2 + 4x - 9 + 3x^2 - x + 12$$

Step 2: Group like terms together:

$$x^2 + 3x^2 + 4x - x - 9 + 12$$

Step 3: Combine the like terms:

$$4x^2 + 3x + 3$$

Example 8 **Subtracting Algebraic Expressions**

Subtract $\left(5x^2 + 4x - 9\right) - \left(3x^2 - x + 12\right)$.

Solution

Step 1: Distribute the negative inside the parentheses (think: $2 - (3 - 4) = 2 - 3 + 4 = -1 + 4 = 3$, which is the correct answer). You cannot just drop the parentheses (for example, $2 - 3 - 4 = -1 - 4 = -5$, which is not correct as we have already verified the answer is 3):

$$5x^2 + 4x - 9 - 3x^2 + x - 12$$

Step 2: Group like terms together:

$$5x^2 - 3x^2 + 4x + x - 9 - 12$$

Step 3: Combine the like terms:

$$2x^2 + x - 21$$

Before looking at multiplying algebraic expressions we look at the **Distributive Property**, which says that to multiply a sum, first you multiply each term in the sum and then you add the products. For example, $5(4 + 3) = 5(4) + 5(3) = 20 + 15 = 35$ can also be solved as $5(4 + 3) = 5(7) = 35$. If we use a variable, then $5(x + 3) = 5x + 15$.

We can extended this example to $(5 + 2)(4 + 3) = (5)(4) + (5)(3) + (2)(4) + (2)(3) = 20 + 15 + 8 + 6 = 49$, which can also be solved as $(5 + 2)(4 + 3) = (7)(7) = 49$. If we use variables, then $(x + 5)(x + 4) = (x)(x) + (x)(4) + (5)(x) + (5)(4) = x^2 + 4x + 5x + 20 = x^2 + 9x + 20$.

FORMULA

Distributive Property: $a(b + c) = ab + ac$

Example 9 **Simplifying an Expression Using the Order of Operations**

Simplify each expression.

1. $(x-3)5$
2. $(-3)(x+y-2)$
3. $5^2(7+3)(x)$
4. $4+x\cdot 5$
5. $(4+x)\cdot 5$

Solution

1. $5x-3\cdot 5=5x-15$
2. $(-3)\cdot x+(-3)\cdot y-(-3)\cdot 2=-3x-3y+6$
3. $25(7+3)(x)=25(10)(x)=250x$
4. $4+5x$
5. $(4)\cdot(5)+(x)\cdot(5)=20+5x$

Example 10 **Multiplying Algebraic Expressions**

Multiply $(4x-9)(x+2)$.

Solution

Step 1: Use the Distributive Property:

$$(4x)(x)+(4x)(2)-(9)(x)-(9)(2)$$

Step 2: Multiply:

$$4x^2+8x-9x-18$$

Step 3: Combine the like terms:

$$4x^2-x-18$$

CHECKPOINT

You may have heard the term FOIL which stands for: First, Outer, Inner, Last. FOIL essentially describes a way to use the Distributive Property if you multiply a two-term expression by another two-term expression, but FOIL only works in that specific situation. For example, suppose you have a two-term expression multiplied by a three-term expression, such as $(x+2)(x+y-5)$*. What terms qualify as inner terms and what terms qualify as outer terms? In this particular situation, FOIL cannot possibly work; the multiplication of* $(x+2)(x+y-5)$ *should yield six terms, where FOIL is designed to only give you four! The Distributive Property works regardless of how many terms there are. FOIL can be misleading and applied inappropriately if not fully understood!*

Example 11 **Dividing Algebraic Expressions**

Divide $\left(8x^2 + 4x - 16\right) \div (4x)$.

Solution

Divide EACH term by $4x$:

$$\left(8x^2 \div 4x\right) + (4x \div 4x) - (16 \div 4x) = 2x + 1 - \frac{4}{x}$$

CHECKPOINT

Be careful how you divide! Sometimes students incorrectly divide only one term on top by the bottom term. For example, $\frac{8x^2+6x-3}{2x}$ *might turn into* $4x + 3x - 3 = 7x - 3$ *if done incorrectly. When we divide expressions, EACH term is divided by the divisor. So,* $\frac{8x^2+6x-3}{2x} = \frac{8x^2}{2x} + \frac{6x}{2x} - \frac{3}{2x} = 4x + 3 - \frac{3}{2x}$*. If you forget, it is always a good idea to check these rules by creating an example using numerical expressions. For example,* $\frac{9+6+3}{3} = \frac{18}{3} = 6$*. Dividing each term on top by 3 would yield* $\frac{9+6+3}{3} = \frac{9}{3} + \frac{6}{3} + \frac{3}{3} = 3 + 2 + 1 = 6$*, which is the correct answer. However, if you just divided the 9 on top by the 3 on the bottom, getting* $\frac{9+6+3}{3} = 3 + 6 + 3 = 12$*, this does not result in the correct answer.*

PEOPLE IN MATHEMATICS

Al-Khwarizmi

Figure 3: Al-Khwarizmi

Abu Ja’far Muhammad ibn Musa Al-Khwarizmi was born around 780 AD, probably in or around the region of Khwarizm, which is now part of modern-day Uzbekistan. For most of his adult life, he worked as a scholar at the House of Wisdom in Baghdad, Iraq. He wrote many mathematical works during his life, but is probably most famous for his book *Al-kitab al-muhtasar fi hisab al-jabr w’al’muqabalah*, which translates to *The Condensed Book on the Calculation of al-Jabr* (*completion*) *and al’muqabalah* (*balancing*). The word *al-jabr* would eventually become the word we use to describe the topic that he was writing about in this book: *algebra*. From another book of his, with the Latin title *Algoritmi de numero Indorum* (*Al-Khwarizmi on the Hindu Art of Reckoning*), our word *algorithm* is derived. In addition to writing on mathematics, Al-Khwarizmi wrote works on astronomy, geography, the sundial, and the calendar.

In 2012, Andrew Hacker wrote an opinion piece in the *New York Times Magazine* suggesting that teaching algebra in high school was a waste of time. Keith Devlin, a British mathematician, was asked to comment on Hacker’s article by his students in his Stanford University Continuing Studies course "Mathematics: Making the Invisible Visible" on iTunes University. Devlin concludes that Hacker was displaying his ignorance of what algebra is.

VIDEO

Q&A: Why We Teach Algebra

Key Terms

- variable
- constant
- expression
- equation
- equal sign
- term
- coefficient
- like terms
- Distributive Property

Key Concepts

- Algebra is useful because it allows us to understand many situations in real life by modeling them with expressions.
- Algebraic expressions are the building blocks of algebra. From algebraic expressions we can create algebraic equations.
- Algebraic expressions are often simplified and evaluated using the four arithmetic operations.

Videos

- Q&A: Why We Teach Algebra

Formulas

- Distributive Property: $a(b + c) = ab + ac$

5.2 Linear Equations in One Variable with Applications

Figure 4: Most gyms have a monthly membership fee.

Learning Objectives

After completing this section, you should be able to:

1. Solve linear equations in one variable using properties of equations.
2. Construct a linear equation to solve applications.
3. Determine equations with no solution or infinitely many solutions.
4. Solve a formula for a given variable.

In this section, we will study linear equations in one variable. There are several real-world scenarios that can be represented by linear equations: taxi rentals with a flat fee and a rate per mile; cell phone bills that charge a monthly fee plus a separate rate per text; gym memberships with a monthly fee plus a rate per class taken; etc. For example, if you join your local gym at \$10 per month and pay \$5 per class, how many classes can you take if your gym budget is \$75 per month?

Linear Equations and Applications

Solving any equation is like discovering the answer to a puzzle. The purpose of solving an equation is to find the value or values of the variable that makes the equation a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle! There are many types of equations that we will learn to solve. In this section, we will focus on a **linear equation**, which is an equation in one variable that can be written as

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$, such that a is the coefficient of x and b is the constant.

To solve a linear equation, it is a good idea to have an overall strategy that can be used to solve any linear equation. In the Example 5.12, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

Example 1 **Solving a Linear Equation Using a General Strategy**

Solve $7(n-3)-8=-15$

Solution

Step 1: Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$
Step 2: Collect all variable terms on one side of the equation.	Nothing to do; all n-terms are on the left side.	$7n-29=-15$
Step 3: Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n-29+29=-15+29$ $7n=14$
Step 4: Make the coefficient of the variable term equal to 1.	Divide each side by 7. Simplify.	$\frac{7n}{7}=\frac{14}{7}$ $n=2$
Step 5: Check the solution.	Let $n=2$ Subtract.	Check: $7(n-3)-8=-15$ $7(2-3)-8\stackrel{?}{=}-15$ $7(-1)-8\stackrel{?}{=}-15$ $-7-8\stackrel{?}{=}-15$ $-15=-15\checkmark$

In Example 1, we used both the addition and division property of equations. All the properties of equations are summarized in table below. Basically, what you do to one side of the equation, you must do to the other side of the equation to preserve equality.

Operation	Property	Example
Addition	If $a = b$ Then $a + c = b + c$	$\begin{aligned} 2 &= 2 \\ 2+3 &= 2+3 \\ 5 &= 5 \end{aligned}$
Subtraction	If $a = b$ Then $a - c = b - c$	$\begin{aligned} 5 &= 5 \\ 5-2 &= 5-2 \\ 3 &= 3 \end{aligned}$
Multiplication	If $a = b$ Then $a \cdot c = b \cdot c$	$\begin{aligned} 3 &= 3 \\ 3 \cdot 4 &= 3 \cdot 4 \\ 12 &= 12 \end{aligned}$
Division	If $a = b$ Then $a \div c = b \div c$ for $c \neq 0$	$\begin{aligned} 8 &= 8 \\ 8 \div 2 &= 8 \div 2 \\ 4 &= 4 \end{aligned}$

CHECKPOINT

Be careful to multiply and divide every term on each side of the equation. For example, $2 + x = \frac{x}{3}$ *is solved by multiplying BOTH sides of the equation by 3 to get* $3(2 + x) = 3\left(\frac{x}{3}\right)$ *which gives* $6 + 3x = x$*. Using parentheses will help you remember to use the distributive property! A division example, such as* $3(x + 2) = 6x + 9$*, can be solved by dividing BOTH sides of the equation by 3 to get* $\frac{3(x+2)}{3} = \frac{6x+9}{3}$*, which then will lead to* $x + 2 = 2x + 3$*.*

Example 2 **Solving a Linear Equation Using Properties of Equations**

Solve $9(y - 2) - y = 16 + 7y$.

Solution

Step 1: Simplify each side.

$$\begin{aligned} 9(y-2) - y &= 16 + 7y \\ 9y - 18 - y &= 16 + 7y \\ 8y - 18 &= 16 + 7y \end{aligned}$$

Step 2: Collect all variables on one side.

$$\begin{aligned} 8y - 18 - 7\boldsymbol{y} &= 16 + 7y - 7\boldsymbol{y} \\ y - 18 &= 16 \end{aligned}$$

Step 3: Collect constant terms on one side.

$$\begin{aligned} y - 18 + 18 &= 16 + 18 \\ y &= 34 \end{aligned}$$

Step 4: Make the coefficient of the variable 1. Already done!

Step 5: Check.

$$\begin{aligned} 9(34) - 18 - (34) &\stackrel{?}{=} 16 + 7(34) \\ 306 - 18 - 34 &\stackrel{?}{=} 16 + 238 \\ 288 - 34 &\stackrel{?}{=} 254 \\ 254 &= 254\checkmark \end{aligned}$$

WHO KNEW?

Who Invented the Symbol for Equals ?

Before the creation of a symbol for equality, it was usually expressed with a word that meant equals, such as *aequales* (Latin), *esgale* (French), or *gleich* (German). Welsh mathematician and physician Robert Recorde is given credit for inventing the modern sign. It first appears in writing in *The Whetstone of Witte*, a book Recorde wrote about algebra, which was published in 1557. In this book, Recorde states, “I will set as I do often in work use, a pair of parallels, or Gemowe (twin) lines of one length, thus: ===, because no two things can be more equal.” Although his version of the sign was a bit longer than the one we use today, his idea stuck and “=” is used throughout the world to indicate equality in mathematics.

In Algebraic Expressions, you translated an English sentence into an equation. In this section, we take that one step further and translate an English paragraph into an equation, and then we solve the equation. We can go back to the opening question in this section: *If you join your local gym at $10 per month and pay $5 per class, how many classes can you take if your gym budget is $75 per month?* We can create an equation for this scenario and then solve the equation (see Example 4).

Example 3 **Constructing a Linear Equation to Solve an Application**

The Beaudrie family has two cats, Basil and Max. Together, they weigh 23 pounds. Basil weighs 16 pounds. How much does Max weigh?

Solution

Let b = Basil’s weight and m = Max’s weight.

$$b + m = 23$$

We also know that Basil weighs 16 pounds so:

Steps 1 and 2: $16 + m = 23$

Since both sides are simplified, the variable is on one side of the equation, we start in Step 3 and collect the constants on one side:

Step 3:

$$\begin{aligned} 16 + m - 16 &= 23 - 16 \\ m &= 7 \end{aligned}$$

Step 4: is already done so we go to Step 5:

Step 5:

$$\begin{aligned} 16 + 7 &\stackrel{?}{=} 23 \\ 23 &= 23 \checkmark \end{aligned}$$

Basil weighs 16 pounds and Max weighs 7 pounds.

Example 4 **Constructing a Linear Equation to Solve Another Application**

If you join your local gym at \$10 per month and pay \$5 per class, how many classes can you take if your gym budget is \$75 per month?

Solution

If we let $x =$ number of classes, the expression $5x + 10$ would represent what you pay per month if each class is \$5 and there's a \$10 monthly fee per class. \$10 is your constant. If you want to know how many classes you can take if you have a \$75 monthly gym budget, set the equation equal to 75. Then solve the equation $5x + 10 = 75$ for x.

Steps 1 and 2:

$$5x + 10 = 75$$

Step 3:

$$\begin{aligned} 5x + 10 - 10 &= 75 - 10 \\ 5x &= 65 \end{aligned}$$

Step 4:

$$\begin{aligned} \frac{5x}{5} &= \frac{65}{5} \\ x &= 13 \end{aligned}$$

Step 5:

$$\begin{aligned} 5(13) + 10 &\stackrel{?}{=} 75 \\ 65 + 10 &\stackrel{?}{=} 75 \\ 75 &= 75 \checkmark \end{aligned}$$

The solution is 13 classes. You can take 13 classes on a \$75 monthly gym budget.

Example 5 **Constructing an Application from a Linear Equation**

Write an application that can be solved using the equation $50x + 35 = 185$. Then solve your application.

Solution

Answers will vary. Let's say you want to rent a snowblower for a huge winter storm coming up. If $x =$ the number of days you rent a snowblower, then the expression $50x + 35$ represents what you pay if, for each day, it costs \$50 to rent the snowblower and there is a \$35 flat rental fee. \$35 is the constant. To find out how many days you can rent a snowblower for \$185, set the expression equal to 185. Then solve the equation $50x + 35 = 185$ for x.

Steps 1 and 2:

$$50x + 35 = 185$$

Step 3:

$$\begin{aligned} 50x + 35 - 35 &= 185 - 35 \\ 50x &= 150 \end{aligned}$$

Step 4:

$$\begin{aligned} \frac{50x}{50} &= \frac{150}{50} \\ x &= 3 \end{aligned}$$

Step 5:

$$\begin{aligned} 50(3) + 35 &\stackrel{?}{=} 185 \\ 150 + 35 &\stackrel{?}{=} 185 \\ 185 &= 185 \checkmark \end{aligned}$$

The equation is $50x + 35 = 185$ and the solution is 3 days. You can rent a snowblower for 3 days on a \$185 budget.

Linear Equations with No Solutions or Infinitely Many Solutions

Every linear equation we have solved thus far has given us one numerical solution. Now we'll look at linear equations for which there are no solutions or infinitely many solutions.

Example 6 **Solving a Linear Equation with No Solution**

Solve $3(x + 4) = 4x + 8 - x$.

Solution

Step 1: Simplify each side. $3(x + 4) = 4x + 8 - x$

$$3x + 12 - 3x = 3x + 8 - 3x$$

Step 2: Collect all variables to one side. $3x + 12 - 3\boldsymbol{x} = 3x + 8 - 3\boldsymbol{x}$

$$12 = 8$$

The variable x disappeared! When this happens, you need to examine what remains. In this particular case, we have $12 = 8$, which is not a true statement. When you have a false statement, then you know the equation has no solution; there does not exist a value for x that can be put into the equation that will make it true.

Example 7 **Solving a Linear Equation with Infinitely Many Solutions**

Solve $2(x + 5) = 4(x + 3) - 2x - 2$.

Solution

Step 1:

$$\begin{aligned} 2(x+5) &= 4(x+3) - 2x - 2 \\ 2x + 10 &= 4x + 12 - 2x - 2 \\ 2x + 10 &= 2x + 10 \end{aligned}$$

Step 2:

$$\begin{aligned} 2x + 10 - 2\boldsymbol{x} &= 2x + 10 - 2\boldsymbol{x} \\ 10 &= 10 \end{aligned}$$

As with the previous example, the variable disappeared. In this case, however, we have a true statement ($10 = 10$). When this occurs we say there are infinitely many solutions; any value for x will make this statement true.

Solving a Formula for a Given Variable

You are probably familiar with some geometry formulas. A **formula** is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be able to manipulate formulas and solve for specific variables.

To solve a formula for a specific variable means to isolate that variable on one side of the equal sign with a coefficient of 1. All other variables and constants are on the other side of the equal sign. To see how to solve a formula for a specific variable, we will start with the distance, rate, and time formula.

Example 8 **Solving for a Given Variable with Distance, Rate, and Time**

Solve the formula $d = rt$ for t. This is the distance formula where d = distance, r = rate, and t = time.

Solution

Divide both sides by r: $\frac{d}{\boldsymbol{r}} = r\frac{t}{\boldsymbol{r}}$

$$\frac{d}{r} = t$$

VIDEO

Solving for a Variable in an Equation

Example 9 **Solving for a Given Variable in the Area Formula for a Triangle**

Solve the formula $A = ½\, bh$ for h. This is the area formula of a triangle where A = area, b = base, and h = height.

Solution

Step 1: Multiply both sides by 2.

$$2A = 2(\tfrac{1}{2}\,bh)$$
$$2A = bh$$

Step 2: Divide both sides by b.

$$\frac{2A}{b} = \frac{bh}{b}$$
$$\frac{2A}{b} = h$$
$$h = \frac{2A}{b}$$

WORK IT OUT

Using Algebra to Understand Card Tricks

You will need to perform this card trick with another person. Before you begin, the two people must first decide which of the two will be the *Dealer* and which will be the *Partner*, as each will do something different. Once you have decided upon that, follow the steps here:

Step 1: Dealer and Partner: Take a regular deck of 52 cards, and remove the face cards and the 10s.

Step 2: Dealer and Partner: Shuffle the remaining cards

Step 3: Dealer and Partner: Select one card each, but keep them face down and don't look at them yet.

Step 4: Dealer: Look at your card (just the Dealer!). Multiply its value by 2 (Aces = 1).

Step 5: Dealer: Add 2 to this result.

Step 6: Dealer: Multiply your answer by 5.

Step 7: Partner: Look at your card.

Step 8: Partner: Calculate: 10 - your card, and tell this information to the dealer.

Step 9: Dealer: Subtract the value the Partner tells you from your total to get a final answer.

Step 10: Dealer: verbally state the final answer.

Step 11: Dealer and Partner: Turn over your cards. Now, answer the following questions

1. Did the trick work? How do you know?
2. Why did this occur? In other words, how does this trick work?

Key Terms

- linear equation

Key Concepts

- Solving linear equations means discovering what the value of the variable in a linear equation represents in the given conditions.
- When solving a linear equation, most often you will have one solution; however, a linear equation may have no solutions or infinitely many solutions.

Videos

- Solving for a Variable in an Equation

5.3 Linear Inequalities in One Variable with Applications

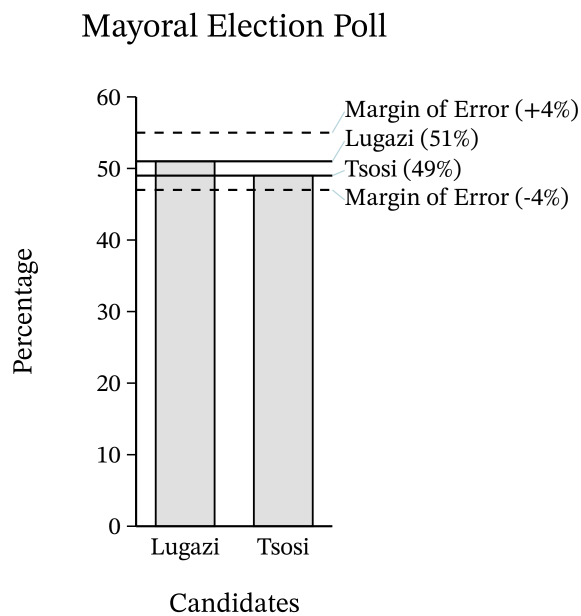


Figure 5: These poll results, showing a margin of error at 4 percent, are an example of a real-world scenario that can be represented by linear inequalities.

Learning Objectives

After completing this section, you should be able to:

1. Graph inequalities in one variable.
2. Solve linear inequalities in one variable.
3. Construct a linear inequality to solve applications.

   In this section, we will study linear inequalities in one variable. Inequalities can be used when the possible values (answers) in a certain situation are numerous, not just a few, or when the exact value (answer) is not known but it is known to be within a range of possible values. There are many real-world scenarios that can be represented by linear inequalities. For example, consider the survey of the mayoral election in Surveys and polls are usually conducted with only a small group of people. The margin of error indicates a range of how the actual group of voters would vote given the results of the survey. This range can be expressed using inequalities.

Another example involves college tuition. Say a local community college charges \$113 per credit hour. You budget \$1,500 for tuition this fall semester. What are the number of credit hours that you could take this fall? Since this answer could be many different values, it can be expressed as an inequality.

Graphing Inequalities on the Number Line

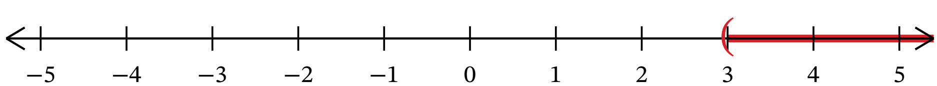
In Algebraic Expressions, we introduced equality and the $=$ symbol. In this section, we look at inequality and the symbols $<$, $>$, $\leq$, and $\geq$. The table below summarizes the symbols and their meaning.

Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Suppose you had the inequality statement $x > 3$. What possible number or numbers would make the inequality $x > 3$ true? If you are thinking, "x could be 4," that's correct, but x could also be 5, 6, 37, 1 million, or even 3.001. The number of solutions is infinite; any number greater than 3 is a solution to the inequality $x > 3$.

Rather than trying to list all possible solutions, we show all the solutions to the inequality $x > 3$ on the number line. All the numbers to the right of 3 on the number line are shaded, to show that all numbers greater than 3 are solutions. At the number 3 itself, an open parenthesis is drawn, since the number 3 is not part of the solutions of $x > 3$.

We can also represent inequalities using interval notation. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "infinity." Infinity is not an actual number. shows both the number line and the interval notation for $x > 3$.



(3, infinity)

Figure 6: The inequality $x > 3$ is graphed on this number line and written in interval notation.

We used the left parenthesis symbol to show that the endpoint of the inequality is not included. Parentheses are used when the endpoints are not included as a possible answer to the inequality. The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

The inequality $x \leq 1$ means all numbers less than or equal to 1. To illustrate that solution on a number line, we first put a bracket at $x = 1$ brackets are used when the endpoint is included. We then shade in all the numbers to the left of 1, to show that all numbers less than one are solutions. There is no lower end to those numbers. We write $x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity." shows both the number line and interval notation for $x = 1$.

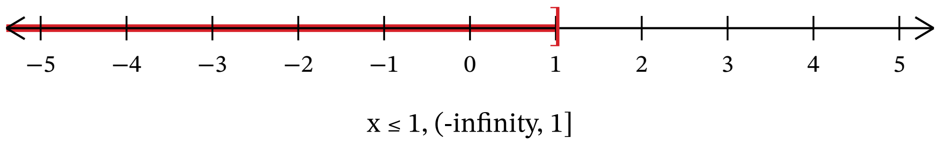


Figure 7: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

summarizes the general representations in both number line form and interval notation of solutions for $x > a$, $x < a$, $x \geq a$, and $x \leq a$.

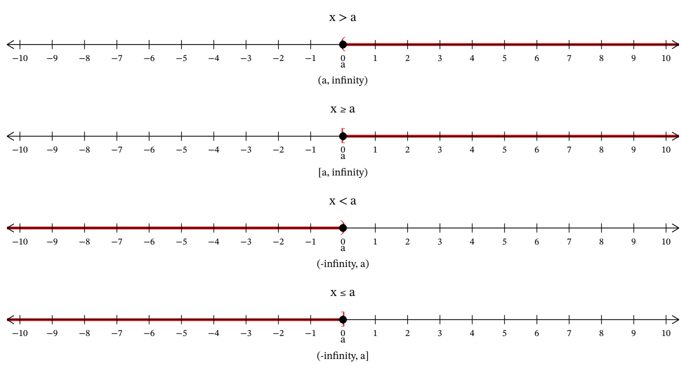


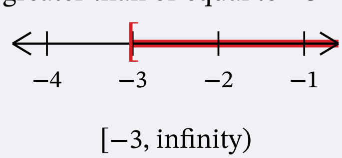
Figure 8: Summary of representations in number line form and interval notation.

Example 1 **Graphing an Inequality**

Graph the inequality $x \geq -3$ and write the solution in interval notation.

Solution

Shade to the right of −3 to show all the numbers greater than −3, and put a bracket at −3 to show that the numbers are greater than or equal to −3



Write in interval notation starting at −3 with a bracket to show that −3 is included in the solution and then infinity because the solution includes all the numbers greater than or equal to −3:

$$[-3, \infty)$$

Example 2 **Graphing a Compound Inequality**

Graph the inequality $x > -3$ and $x < 4$ and write the solution in interval notation.

Solution

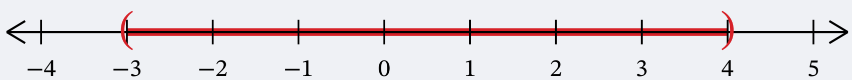
Step 1: Graph $x > -3$.

(-3, 4)

Step 2: Graph $x < 4$.

(-infinity, 4)

Step 3: Graph both on the same number line and think of where the solutions are to BOTH inequalities . This will be where BOTH are shaded.



(-3, 4)

Step 4: Write the solution in interval notation:

$$(-3, 4)$$

WHO KNEW?

Where Did the Inequality Symbols Come From?

The first use of the $<$ symbol to represent “less than” and $>$ to represent “greater than” appeared in a mathematics book written by Englishman Thomas Harriot that was published in 1631. However, Harriot did not invent the symbols…the editor of the book did! Harriot used triangular symbols to represent less than and greater than; the editor, for reasons unknown, changed to symbols that are similar to the ones we use today. The symbols used to represent less than or equal to, and greater than or equal to ($\leq$ and $\geq$) were first used in 1731 by French hydrologist and surveyor Pierre Bouguer. Interestingly, English mathematician John Wallis had used similar symbols as early as 1670, but he put the bar above the less than and greater than symbols instead of below them.

Solving Linear Inequalities

A **linear inequality** is much like a linear equation—but the equal sign is replaced with an inequality sign. A linear inequality is an inequality in one variable that can be written in one of the forms $ax + b < c, ax + b \leq c, ax + b \geq c$, or $ax + b > c$, where a, b, and c are all real numbers.

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities. We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example, we know that 2 is less than 4, i.e., $2 < 4$. If we add 6 to both sides of this inequality, we still have a true statement:

$$\begin{aligned} 2+6 &< 4+6 \\ 8 &< \quad 10 \end{aligned}$$

The same would happen if we subtracted 6 from both sides of the inequality; the statement would stay true:

$$\begin{aligned} 2-6 &< 4-6 \\ -4 &< \quad -2 \end{aligned}$$

Notice that the inequality signs stayed the same. This leads us to the Addition and Subtraction Properties of Inequality.

FORMULA

For any numbers a, b, and c, if $a < b$, then $a + c < b + c$ and $a - c < b - c$.

For any numbers a, b, and c, if $a > b$, then $a + c > b + c$ and $a - c > b - c$.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality the same. But what happens to an inequality when we divide or multiply both sides by a number? Let's first multiply and divide both sides by a positive number, starting with an inequality we know is true, $10 < 15$. We will multiply and divide this inequality by 5:

$$\begin{array}{ccc} 10 < & 1510 & < 15 \\ 10(5)? & 15(5)\dfrac{10}{5} & ?\,\dfrac{15}{5} \\ 50? & 752 & ?\,3 \\ 50 < 75(\text{true})2 & & < 3(\text{true}) \end{array}$$

The inequality signs stayed the same. Does the inequality stay the same when we divide or multiply by a negative number? Let's use our inequality $10 < 15$ to find out, multiplying it and dividing it by -5:

$$\begin{array}{ccc} 10 < & 1510 & < 15 \\ 10(-5)? & 15(-5)\dfrac{10}{-5} & ?\,\dfrac{15}{-5} \\ -50? & -75-2 & ?\,-3 \\ -50 > -75(\text{true}) - 2 > -3(\text{true}) & & \end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction in order to make it true! To summarize, when we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses. This gives us the Multiplication and Division Property of Inequality.

FORMULA

For any numbers a, b, and c,

multiply or divide by a positive:	if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$
	if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
multiply or divide by a negative:	if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
	if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

To summarize, when we divide or multiply an inequality by:

- a positive number, the inequality sign stays the same.
- a negative number, the inequality sign reverses.

CHECKPOINT

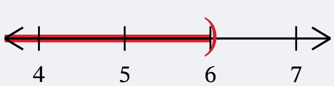
Be careful to only reverse the inequality sign when you are multiplying and dividing by a negative. You do NOT reverse the inequality sign when you add or subtract a negative. For example, $2x < -4$ is solved by dividing both sides of the inequality by 2 to get $x < -2$. You do NOT reverse the inequality sign because there is a negative 4. As another example, $-2x + 5 < 3x$ is solved by adding $-2x$ to both sides to get $5 < 5x$. This does not reverse the inequality sign because we were not multiplying or dividing by a negative. We then divide both sides by 5 and get $1 < x$.

Example 3 **Solving a Linear Inequality Using One Operation**

Solve $9y < 54$, graph the solution on the number line, and write the solution in interval notation.

Solution

$$\begin{aligned} 9y &< 54 \\ \frac{9y}{9} &< \frac{54}{9} \\ y &< 6 \end{aligned}$$



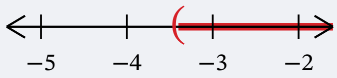
(-infinity, 6)

Example 4 **Solving a Linear Inequality Using Multiple Operations**

Solve the inequality $6y \le 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

Solution

$$\begin{aligned} 6y &\leq 11y + 17 \\ 6y - 11\mathbf{y} &\leq 11\mathbf{y} + 17 - 11y \\ -5y &\leq 17 \\ \frac{-5y}{-5} &\geq -\frac{17}{5} \\ y &\geq -\frac{17}{5} \end{aligned}$$



(-17/5, infinity)

Solving Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with **linear inequalities** is very much like the one we used when we solved applications with equations. We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Example 5 **Constructing a Linear Inequality to Solve an Application with Tablet Computers**

A teacher won a mini grant of $4,000 to buy tablet computers for their classroom. The tablets they would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets the teacher can buy?

Solution

Let $t =$ the number of tablets .

t times $254.12 has to be less than $4,000, so $254.12t \leq 4,000$.

Solve for t:

$$\begin{aligned} \frac{254.12t}{254.12} &\leq \frac{4,000}{254.12} \\ t &\leq 15.74 \end{aligned}$$

The teacher can buy 15 tablets and stay under $4,000.

Example 6 **Constructing a Linear Inequality to Solve a Tuition Application**

The local community college charges $113 per credit hour. Your budget is $1,500 for tuition this fall semester. What number of credit hours could you take this fall?

Solution

Let $c =$ the number of credit hours you could take.

c times $113 has to be less than $1,500, so $113c \leq 1,500$.

Solve for c:

$$\begin{aligned} \frac{113c}{113} &\leq \frac{1500}{113} \\ c &\leq 13.27 \end{aligned}$$

You can take up to 13 credits and stay under $1,500.

Example 7 **Constructing a Linear Inequality to Solve an Application with Travel Costs**

Brenda's best friend is having a destination wedding and the event will last 3 days and 3 nights. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 for airfare, $375 for food and entertainment, and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Solution

Let $b =$ number of babysitting hours.

b times $15 plus $500 has to be more than $\$350 + \$375 + \$\frac{60}{\text{night}}$, so $15b + 500 \geq 350 + 375 + 60(3)$.

Solve for b:

$$\begin{aligned} 15b + 500 - 500 &\geq 905 - 500 \\ 15b &\geq 405 \\ \frac{15b}{15} &\geq \frac{405}{15} \\ b &\geq 27 \end{aligned}$$

Brenda must babysit at least 27 hours.

TECH CHECK

The Desmos activities called "Inequalities on a Number Line" and "Compound Inequalities on a Number Line" are ways for students to develop and deepen their understanding of inequalities. Teachers will need a Desmos account to assign the activity for student use. Once they have assigned the activity to their students, teachers need to share the code for the activity with their students. Students will input the code to work on the activity.

Key Terms

- linear inequality
- Addition and Subtraction Property of Linear Inequalities
- Multiplication and Division Property of Linear Inequalities

Key Concepts

- Inequalities can be used when the possible values (answers) in a certain situation are numerous, or when the exact value (answer) is not known, but it is known to be within a range of possible values.
- Linear inequalities can be represented using a number line or using interval notation.

Formulas

- For any numbers a, b, and , if $a < b$, then $a + c < b + c$ and $a - c < b - c$.
- For any numbers a, b, and c, if $a > b$, then $a + c > b + c$ and $a - c > b - c$.
- For any numbers a, b, and c,
  multiply or divide by a positive:
  if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$
  if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
  multiply or divide by a negative:
  if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
  if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

5.4 Ratios and Proportions

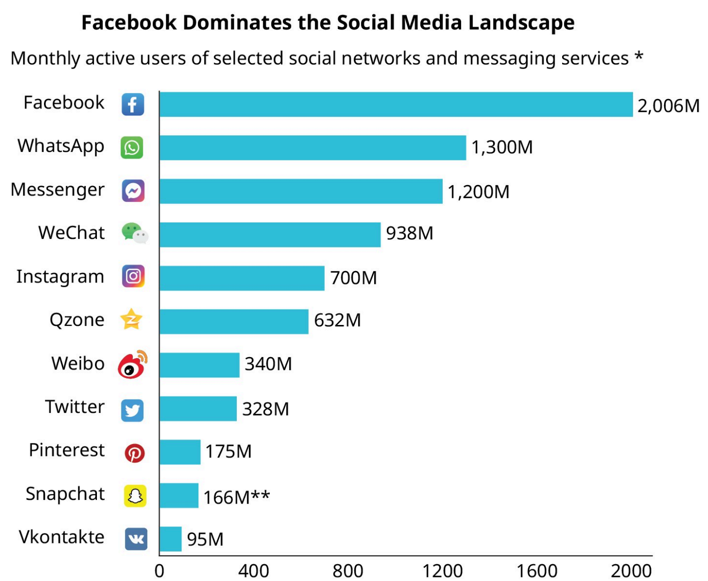


Figure 15: This bar graph shows popular social media app usage. (Source)

Learning Objectives

After completing this section, you should be able to:

1. Construct ratios to express comparison of two quantities.
2. Use and apply proportional relationships to solve problems.
3. Determine and apply a constant of proportionality.
4. Use proportions to solve scaling problems.

   Ratios and proportions are used in a wide variety of situations to make comparisons. For example, using the information from , we can see that the number of Facebook users compared to the number of Twitter users is 2,006 M to 328 M. Note that the "M" stands for million, so 2,006 million is actually 2,006,000,000 and 328 million is 328,000,000. Similarly, the number of Qzone users compared to the number of Pinterest users is in a ratio of 632 million to 175 million. These types of comparisons are ratios.

Constructing Ratios to Express Comparison of Two Quantities

Note there are three different ways to write a **ratio**, which is a comparison of two numbers that can be written as: a to b OR $a : b$ OR the fraction $\frac{a}{b}$. Which method you use often depends upon the situation. For the most part, we will want to write our ratios using the fraction notation. Note that, while all ratios are fractions, not all fractions are ratios. Ratios make part to part, part to whole, and whole to part comparisons. Fractions make part to whole comparisons only.

Example 1 **Expressing the Relationship between Two Currencies as a Ratio**

The Euro (€) is the most common currency used in Europe. Twenty-two nations, including Italy, France, Germany, Spain, Portugal, and the Netherlands use it. On June 9, 2021, 1 U.S. dollar was worth 0.82 Euros. Write this comparison as a ratio.

Solution

Using the definition of ratio, let $a = 1$ U.S. dollar and let $b = 0.82$ Euros. Then the ratio can be written as either 1 to 0.82; or 1:0.82; or $\frac{1}{0.82}$.

Example 2 **Expressing the Relationship between Two Weights as a Ratio**

The gravitational pull on various planetary bodies in our solar system varies. Because weight is the force of gravity acting upon a mass, the weights of objects is different on various planetary bodies than they are on Earth. For example, a person who weighs 200 pounds on Earth would weigh only 33 pounds on the moon! Write this comparison as a ratio.

Solution

Using the definition of ratio, let $a = 200$ pounds on Earth and let $b = 33$ pounds on the moon. Then the ratio can be written as either 200 to 33; or 200:33; or $\frac{200}{33}$.

Using and Applying Proportional Relationships to Solve Problems

Using **proportions** to solve problems is a very useful method. It is usually used when you know three parts of the proportion, and one part is unknown. Proportions are often solved by setting up like ratios. If $\frac{a}{b}$ and $\frac{c}{d}$ are two ratios such that $\frac{a}{b} = \frac{c}{d}$, then the fractions are said to be **proportional**. Also, two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are proportional $\left(\frac{a}{b} = \frac{c}{d}\right)$ if and only if $a \times d = b \times c$.

Example 3 **Solving a Proportion Involving Two Currencies**

You are going to take a trip to France. You have \$520 U.S. dollars that you wish to convert to Euros. You know that 1 U.S. dollar is worth 0.82 Euros. How much money in Euros can you get in exchange for \$520?

Solution

Step 1: Set up the two ratios into a proportion; let x be the variable that represents the unknown. Notice that U.S. dollar amounts are in both numerators and Euro amounts are in both denominators.

$$\frac{1}{0.82} = \frac{520}{x}$$

Step 2: Cross multiply, since the ratios $\frac{a}{b}$ and $\frac{c}{d}$ are proportional, then $a \times d = b \times c$.

$$\begin{aligned} 520(0.82) &= 1(x) \\ 426.4 &= \quad x \end{aligned}$$

You should receive 426.4 Euros (426.4 €).

Example 4 **Solving a Proportion Involving Weights on Different Planets**

A person who weighs 170 pounds on Earth would weigh 64 pounds on Mars. How much would a typical racehorse (1,000 pounds) weigh on Mars? Round your answer to the nearest tenth.

Solution

Step 1: Set up the two ratios into a proportion. Notice the Earth weights are both in the numerator and the Mars weights are both in the denominator.

$$\frac{170}{64} = \frac{1{,}000}{x}$$

Step 2: Cross multiply, and then divide to solve.

$$\begin{aligned} 170x &= 1,000(64) \\ 170x &= \quad 64,000 \\ \frac{170x}{170} &= \quad \frac{64{,}000}{170} \\ x &= \quad 376.5 \end{aligned}$$

So the 1,000-pound horse would weigh about 376.5 pounds on Mars.

Example 5 **Solving a Proportion Involving Baking**

A cookie recipe needs $2\frac{1}{4}$ cups of flour to make 60 cookies. Jackie is baking cookies for a large fundraiser; she is told she needs to bake 1,020 cookies! How many cups of flour will she need?

Solution

Step 1: Set up the two ratios into a proportion. Notice that the cups of flour are both in the numerator and the amounts of cookies are both in the denominator. To make the calculations more efficient, the cups of flour $\left(2\frac{1}{4}\right)$ is converted to a decimal number (2.25).

$$\frac{2.25}{60} = \frac{x}{1020}$$

Step 2: Cross multiply, and then simplify to solve.

$$\begin{aligned} 2.25(1,020) &= 60x \\ 2,295 &= 60x \\ 38.25 &= \quad x \end{aligned}$$

Jackie will need 38.25, or $38\frac{1}{4}$, cups of flour to bake 1,020 cookies.

CHECKPOINT

Part of the definition of proportion states that two fractions $\frac{a}{b}$ *and* $\frac{c}{d}$ *are proportional if* $a \times d = b \times c$. *This is the "cross multiplication" rule that students often use (and unfortunately, often use incorrectly). The only time cross multiplication can be used is if you have two ratios (and only two ratios) set up in a proportion. For example, you cannot use cross multiplication to solve for x in an equation such as* $\frac{2}{5} = \frac{x}{8} + 3x$ *because you do not have just the two ratios. Of course, you could use the rules of algebra to change it to be just two ratios and then you could use cross multiplication, but in its present form, cross multiplication cannot be used.*

PEOPLE IN MATHEMATICS

Eudoxus was born around 408 BCE in Cnidus (now known as Knidos) in modern-day Turkey. As a young man, he traveled to Italy to study under Archytas, one of the followers of Pythagoras. He also traveled to Athens to hear lectures by Plato and to Egypt to study astronomy. He eventually founded a school and had many students.

Eudoxus made many contributions to the field of mathematics. In mathematics, he is probably best known for his work with the idea of proportions. He created a definition of proportions that allowed for the comparison of any numbers, even irrational ones. His definition concerning the equality of ratios was similar to the idea of cross multiplying that is used today. From his work on proportions, he devised what could be described as a method of integration, roughly 2000 years before calculus (which includes integration) would be fully developed by Isaac Newton and Gottfried Leibniz. Through this technique, Eudoxus became the first person to rigorously prove various theorems involving the volumes of certain objects. He also developed a planetary theory, made a sundial still usable today, and wrote a seven volume book on geography called *Tour of the Earth*, in which he wrote about all the civilizations on the Earth, and their political systems, that were known at the time. While this book has been lost to history, over 100 references to it by different ancient writers attest to its usefulness and popularity.

Determining and Applying a Constant of Proportionality

In the last example, we were given that $2\frac{1}{4}$ cups of flour could make 60 cookies; we then calculated that $38\frac{1}{4}$ cups of flour would make 1,020 cookies, and 720 cookies could be made from 27 cups of flour. Each of those three ratios is written as a fraction below (with the fractions converted to decimals). What happens if you divide the numerator by the denominator in each?

$$\frac{2.25}{60} = 0.0375 \quad \frac{38.25}{1{,}020} = 0.0375 \quad \frac{27}{720} = 0.0375\ .$$

The quotients in each are exactly the same! This number, determined from the ratio of cups of flour to cookies, is called the **constant of proportionality**. If the values a and b are related by the equality $\frac{a}{b} = k$, then k is the constant of proportionality between a and b. Note since $\frac{a}{b} = k$, then $b = \frac{a}{k}$. and $b = \frac{a}{k}$.

One piece of information that we can derive from the constant of proportionality is a unit rate. In our example (cups of flour divided by cookies), the constant of proportionality is telling us that it takes 0.0375 cups of flour to make one cookie. What if we had performed the calculation the other way (cookies divided by cups of flour)?

$$\frac{60}{2.25} = 26.66666\ldots \quad \frac{1{,}020}{38.25} = 26.66666\ldots \quad \frac{720}{27} = 26.66666\ldots$$

In this case, the constant of proportionality $\left(26.66666\ldots = 26\frac{2}{3}\right)$ is telling us that $26\frac{2}{3}$ cookies can be made with one cup of flour. Notice in both cases, the "one" unit is associated with the denominator. The constant of proportionality is also useful in calculations if you only know one part of the ratio and wish to find the other.

Example 6 **Finding a Constant of Proportionality**

Isabelle has a part-time job. She kept track of her pay and the number of hours she worked on four different days, and recorded it in the table below. What is the constant of proportionality, or pay divided by hours? What does the constant of proportionality tell you in this situation?

Pay	\$87.50	\$50.00	\$37.50	\$100.00
Hours	7	4	3	8

Solution

To find the constant of proportionality, divide the pay by hours using the information from any of the four columns. For example, $\frac{87.5}{7} = 12.5$. The constant of proportionality is 12.5, or \$12.50. This tells you Isabelle's hourly pay: For every hour she works, she gets paid \$12.50.

Example 7 **Applying a Constant of Proportionality: Running mph**

Zac runs at a constant speed: 4 miles per hour (mph). One day, Zac left his house at exactly noon (12:00 PM) to begin running; when he returned, his clock said 4:30 PM. How many miles did he run?

Solution

The constant of proportionality in this problem is 4 miles per hour (or 4 miles in 1 hour). Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\frac{a \text{ miles}}{b \text{ hours}} = k$$

$\frac{a}{4.5} = 4$ (30 minutes is ½, or 0.5, hours)

$a = 4(4.5)$, since from the definition we know $a = kb$

$$a = 18$$

Zac ran 18 miles.

Example 8 **Applying a Constant of Proportionality: Filling Buckets**

Joe had a job where every time he filled a bucket with dirt, he was paid \$2.50. One day Joe was paid \$337.50. How many buckets did he fill that day?

Solution

The constant of proportionality in this situation is \$2.50 per bucket (or \$2.50 for 1 bucket). Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\frac{a \text{ dollars}}{b \text{ buckets}} = k$$
$$\frac{337.50}{b} = 2.50$$

Since we are solving for b, and we know from the definition that $b = \frac{a}{k}$:

$$b = \frac{337.50}{2.50}$$
$$b = 135$$

Joe filled 135 buckets.

Example 9 **Applying a Constant of Proportionality: Miles vs. Kilometers**

While driving in Canada, Mabel quickly noticed the distances on the road signs were in kilometers, not miles. She knew the constant of proportionality for converting kilometers to miles was about 0.62—that is, there are about 0.62 miles in 1 kilometer. If the last road sign she saw stated that Montreal is 104 kilometers away, about how many more miles does Mabel have to drive? Round your answer to the nearest tenth.

Solution

The constant of proportionality in this situation is 0.62 miles per 1 kilometer. Since $\frac{a}{b} = k$, where k is the constant of proportionality, we have

$$\begin{aligned} \frac{a \text{ miles}}{b \text{ kilometers}} &= k \\ \frac{a}{104} &= 0.62 \\ a &= 0.62(104) \\ a &= 64.48 \end{aligned}$$

Rounding the answer to the nearest tenth, Mabel has to drive 64.5 miles.

Using Proportions to Solve Scaling Problems

Figure 16: A map of the northeastern United States

Ratio and proportions are used to solve problems involving **scale**. One common place you see a scale is on a map (as represented in). In this image, 1 inch is equal to 200 miles. This is the scale. This means that 1 inch on the map corresponds to 200 miles on the surface of Earth. Another place where scales are used is with models: model cars, trucks, airplanes, trains, and so on. A common ratio given for model cars is 1:24—that means that 1 inch in length on the model car is equal to 24 inches (2 feet) on an actual automobile. Although these are two common places that scale is used, it is used in a variety of other ways as well.

Example 10 **Solving a Scaling Problem Involving Maps**

is an outline map of the state of Colorado and its counties. If the distance of the southern border is 380 miles, determine the scale (i.e., 1 inch = how many miles). Then use that scale to determine the approximate lengths of the other borders of the state of Colorado.

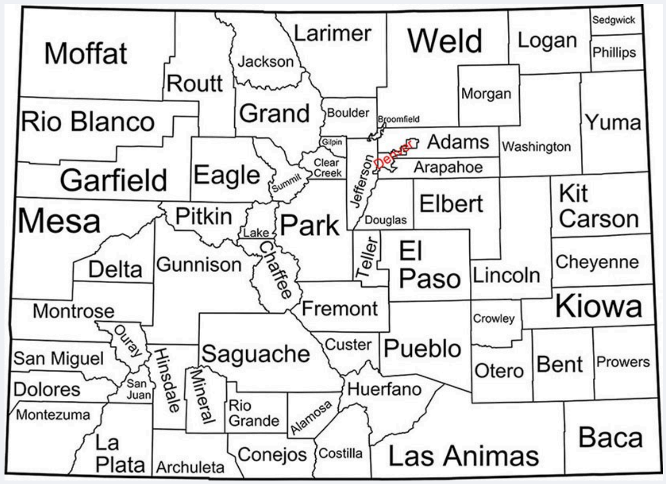


Figure 17: Outline Map of Colorado

Solution

When the southern border is measured with a ruler, the length is 4 inches. Since the length of the border in real life is 380 miles, our scale is 1 inch = 95 miles.

The eastern and western borders both measure 3 inches, so their lengths are about 285 miles. The northern border measures the same as the southern border, so it has a length of 380 miles.

Example 11 **Solving a Scaling Problem Involving Model Cars**

Die-cast NASCAR model cars are said to be built on a scale of 1:24 when compared to the actual car. If a model car is 9 inches long, how long is a real NASCAR automobile? Write your answer in feet.

Solution

The scale tells us that 1 inch of the model car is equal to 24 inches (2 feet) on the real automobile. So set up the two ratios into a proportion. Notice that the model lengths are both in the numerator and the NASCAR automobile lengths are both in the denominator.

$$\begin{aligned} \frac{1}{24} &= \frac{9}{x} \\ 24(9) &= x \\ 216 &= x \end{aligned}$$

This amount (216) is in inches. To convert to feet, divide by 12, because there are 12 inches in a foot (this conversion from inches to feet is really another proportion!). The final answer is:

$$\frac{216}{12} = 18$$

The NASCAR automobile is 18 feet long.

Key Terms

- ratio
- proportion
- constant of proportionality
- scale
- construct ratios
- solve proportions
- use proportions to solve scaling problems

Key Concepts

- A ratio is a comparison of two numbers. The ratio of two numbers a and b can be written as: a to b OR a:b OR the fraction a/b.
- All fractions are ratios, but not all ratios are fractions. Ratios make part to part, part to whole, and whole to part comparisons. Fractions make part to whole comparisons only.
- When two ratios are equal, we say they are in proportion or are proportional.
- Setting up proportions allows us to solve many various situations where three of the four values of the proportion are known.

5.5 Graphing Linear Equations and Inequalities

Figure 18: How much would it cost to fill up your gas tank?

Learning Objectives

After completing this section, you should be able to:

1. Graph linear equations and inequalities in two variables.
2. Solve applications of linear equations and inequalities.

In this section, we will learn how to graph linear equations and inequalities. There are several real-world scenarios that can be represented by graphs of linear inequalities. Think of filling your car up with gasoline. If gasoline is $3.99 per gallon and you put 10 gallons in your car, you will pay $39.90. Your friend buys 15 gallons of gasoline and pays $59.85. You can plot these points on a coordinate system and connect the points with a line to create the graph of a line. You'll learn to do both in this section.

Plotting Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a rectangular coordinate system. The rectangular coordinate system is also called the xy-plane or the "coordinate plane."

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the x-axis. The vertical number line is called the y-axis. These axes divide a plane into four regions, called quadrants. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise.

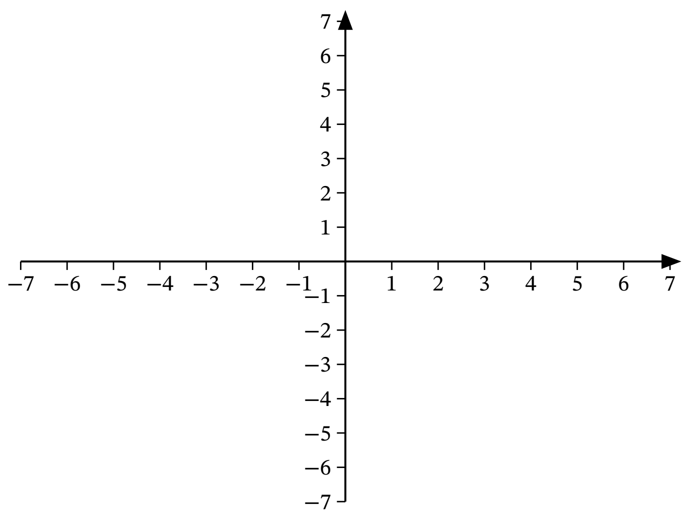


Figure 19: Quadrants on the Coordinate Plane

In the rectangular coordinate system, every point is represented by an **ordered pair** . The first number in the ordered pair is the x-coordinate of the point, and the second number is the y -coordinate of the point. The phrase "ordered pair" means that the order is important. At the point where the axes cross and where both coordinates are zero, the ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

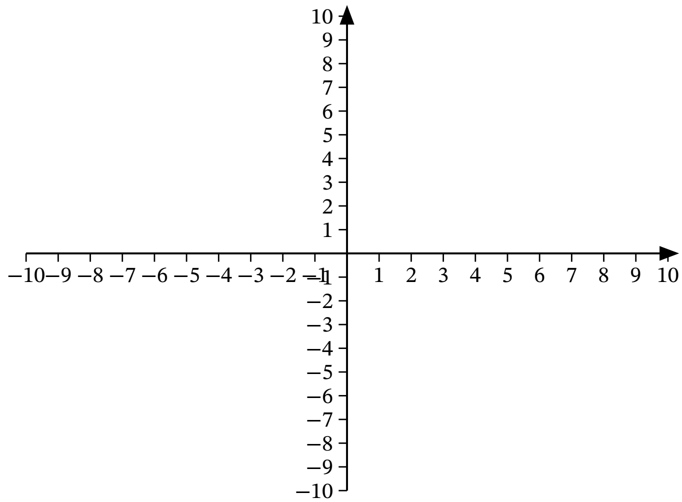


Figure 20: Ordered Pair

We use the coordinates to locate a point on the xy-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the x-axis and lightly sketch a vertical line through $x = 1$. Then, locate 3 on the y-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$.

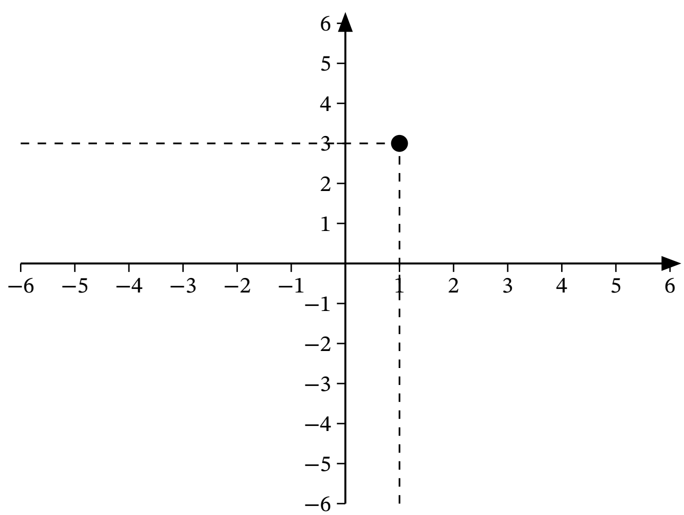


Figure 21: Point $(1, 3)$ Plotted on the Coordinate Plane

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. The dotted lines are just used to help us locate the point $(1, 3)$. When one of the coordinates is zero, the point lies on one of the axes. In , the point $(0, 4)$ is on the y-axis and the point $(-2, 0)$ is on the x-axis.

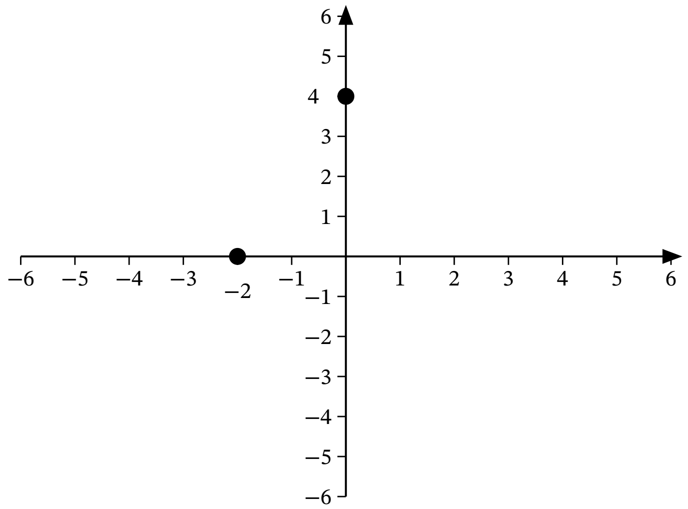


Figure 22: Points $(-2, 0)$ and $(0, 4)$ Plotted on the Coordinate Plane

Example 1 **Plotting Points on a Coordinate System**

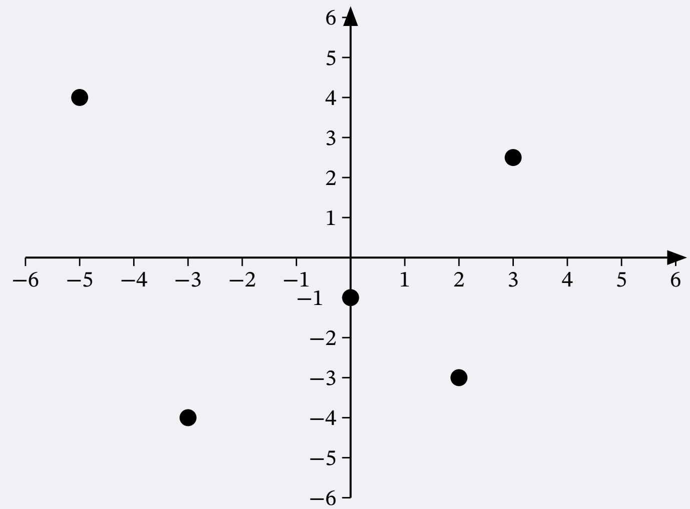
Plot the following points in the rectangular coordinate system and identify the quadrant in which the point is located:

1. $(-5, 4)$
2. $(-3, -4)$
3. $(2, -3)$
4. $(0, -1)$
5. $\left(3, \frac{5}{2}\right)$

Solution

The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate. To plot each point, sketch a vertical line through the x-coordinate and a horizontal line through the y-coordinate . Their intersection is the point.

1. Since $x = -5$, the point is to the left of the y-axis. Also, since $y = 4$, the point is above the x-axis. The point $(-5, 4)$ is in quadrant II.
2. Since $x = -3$, the point is to the left of the y-axis. Also, since $y = -4$, the point is below the x-axis. The point $(-3, -4)$ is in quadrant III.
3. Since $x = 2$, the point is to the right of the y-axis. Since $y = -3$, the point is below the x-axis. The point $(2, -3)$ is in quadrant IV.
4. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the y-axis.
5. Since $x = 3$, the point is to the right of the y-axis. Since $y = \frac{5}{2}$, which is equal to 2.5, the point is above the x-axis. The point $\left(3, \frac{5}{2}\right)$ is in quadrant I.



Graphing Linear Equations in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation, you got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. An equation of this form, where A and B are both not zero, is called a **linear equation in two variables**. Here is an example of a linear equation in two variables, x and y.

$$\boldsymbol{A}x + \boldsymbol{B}y = \boldsymbol{C}$$
$$\boldsymbol{A} + 4y = 8$$
$$\boldsymbol{A} = 1, \boldsymbol{B} = 4, \boldsymbol{C} = 8$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the addition property of equality and rewrite it in $Ax + By = C$ form.

Step 1: Add $3x$ to both sides. $y + 3x = -3x + 5 + 3x$

Step 2: Simplify. $y + 3x = 5$

Step 3: Put it in $Ax + By = C$ form. $3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in **standard form of a linear equation**. Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a **solution** to the linear equation and is represented by the ordered pair (x,y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

We can plot these solutions in the rectangular coordinate system. The points will line up perfectly in a straight line. We connect the points with a straight line to get the graph of the linear equation. We put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of a linear equation. The line shows you *all* the solutions to that linear equation. Every point on the line is a solution of that linear equation. And every solution of the linear equation is on this line. This line is called the graph of the equation. Points *not* on the line are not solutions! The graph of a linear equation $Ax + By = C$ is a straight line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example 2 **Determining Points on a Line**

is the graph of $y = 2x - 3$.

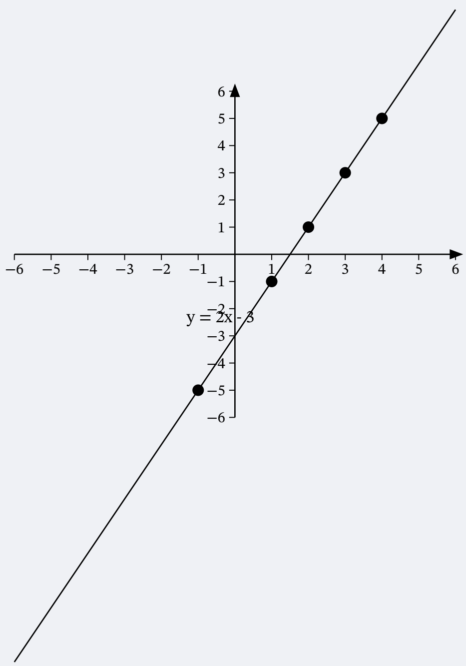


Figure 24: Graph of $y = 2x - 3$

For each ordered pair, decide:

1. Is the ordered pair a solution to the equation?
2. Is the point on the line?

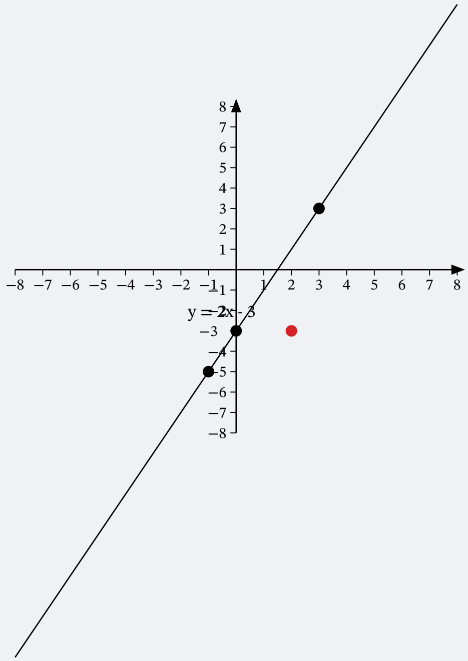
A: $(0, -3)$ B: $(3, 3)$ C: $(2, -3)$ D: $(-1, -5)$

Solution

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

1.
$$\begin{aligned} &\text{A}:(0,-3)\\ y &= 2x-3\\ -3 &\stackrel{?}{=} 2(0)-3\\ -3 &= -3\checkmark \end{aligned}$$
$(0,-3)$ is a solution.
$$\begin{aligned} &\text{B}:(3,3)\\ y &= 2x-3\\ 3 &\stackrel{?}{=} 2(3)-3\\ 3 &= 3\checkmark \end{aligned}$$
$(3,3)$ is a solution.
$$\begin{aligned} &\text{C}:(2,-3)\\ y &= 2x-3\\ -3 &\stackrel{?}{=} 2(2)-3\\ -3 &\neq 1 \end{aligned}$$
$(2,-3)$ is not a solution.
$$\begin{aligned} &\text{D}:(-1,-5)\\ y &= 2x-3\\ -5 &\stackrel{?}{=} 2(-1)-3\\ -5 &= -5\checkmark \end{aligned}$$
$(-1,-5)$ is a solution.

2. Plot the points $(0,-3)$, $(3,3)$, $(2,-3)$, and $(-1,-5)$.



In , the points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line. The points that are solutions to $y = 2x - 3$ are on the line, but the point that is not a solution is not on the line.

The steps to take when graphing a linear equation by plotting points are:

Step 1: Find three points whose coordinates are solutions to the equation. Organize them in a table.

Step 2: Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.

Step 3: Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line, but it will not represent the solutions to the equation. It will be the wrong line. If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong, and you need to check your work.

Example 3 **Graphing a Line by Plotting Points**

Graph the equation: $y = \frac{1}{2}x + 3$.

Solution

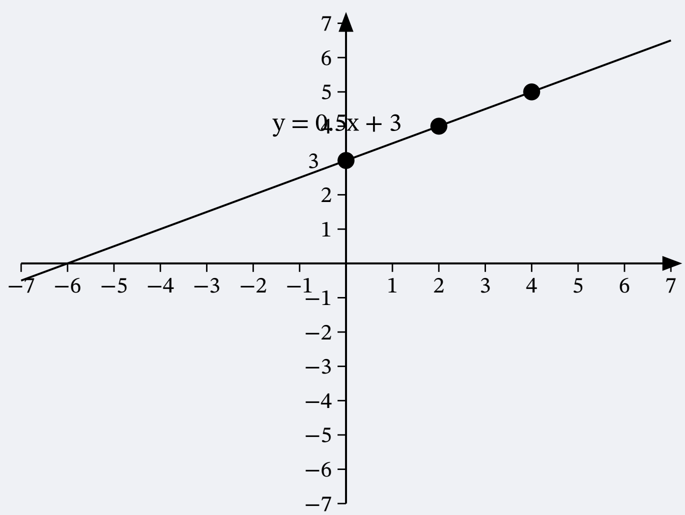
Find three points that are solutions to the equation. Since this equation has the fraction

$\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2, the multiplication by $\frac{1}{2}$ simplifies to a whole number.

$$x = 0x = 2 \qquad x = 4$$
$$y = \frac{1}{2}x + 3y = \frac{1}{2}x + 3 \qquad y = \frac{1}{2}x + 3$$
$$y = \frac{1}{2}(0) + 3y = \frac{1}{2}(2) + 3y = \frac{1}{2}(4) + 3$$
$$y = 0 + 3y = 1 + 3 \qquad y = 2 + 3$$
$$y = 3y = 4 \qquad y = 5$$

x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

Plot the points, check that they line up, and draw the line .



Solving Applications Using Linear Equations in Two Variables

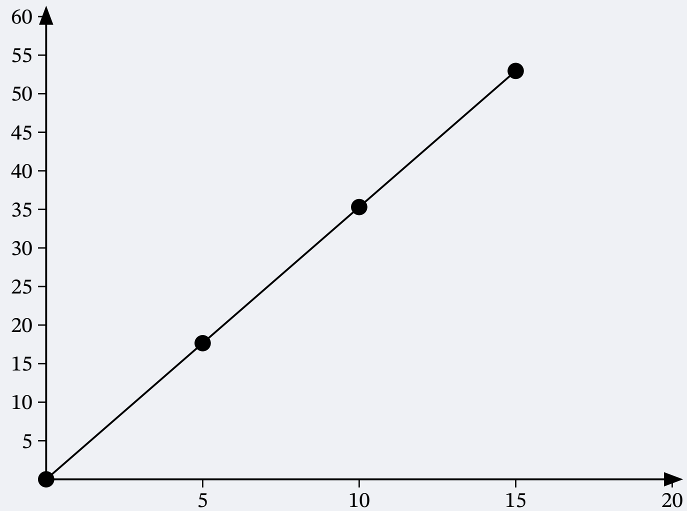
Many fields use linear equalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

Example 4 **Pumping Gas**

Gasoline costs \$3.53 per gallon. You put 10 gallons of gasoline in your car, and pay \$35.30. Your friend puts 15 gallons of gasoline in their car and pays \$52.95. Your neighbor needs 5 gallons of gasoline, how much will they pay?

Solution

Let x = the number of gallons of gasoline and let y = the total cost. If gas is \$3.53 per gallon, then $y = 3.53x$. The two points given are (10, 35.30) and (15, 52.95). Plot the points, check that they line up, and draw the line .



We can see the point at $x = 5$. The y-value is found by multiplying 5 by \$3.53 to get \$17.65. Your neighbor will pay \$17.65.

PEOPLE IN MATHEMATICS

René Descartes

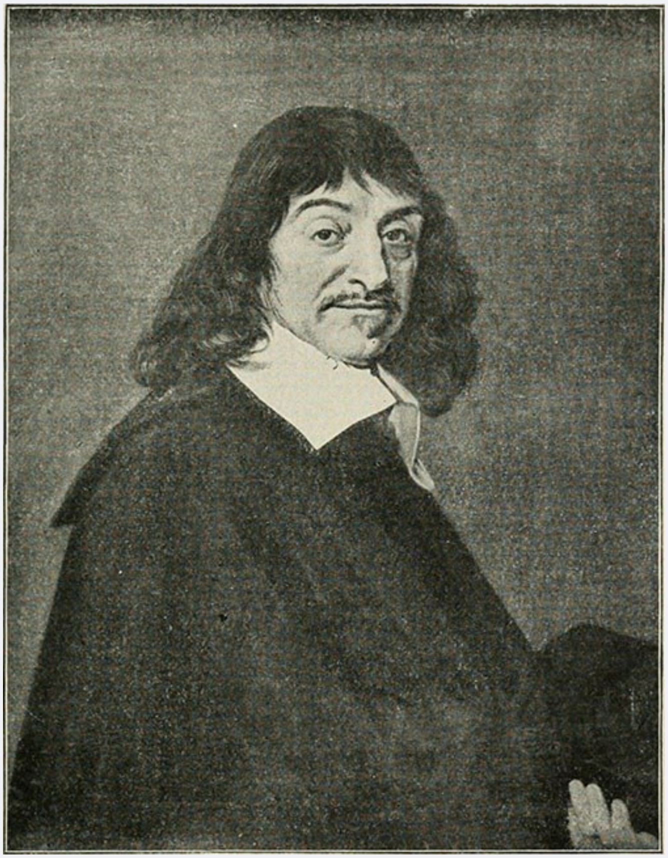


Figure 28: René Descartes

René Descartes was born in 1596 in La Haye, France. He was sickly as a child, so much so that he was allowed to stay in bed until 11:00 AM rather than get up at 5:00 AM like the other school children. He kept this habit of rising late for most of the rest of his life.

After his primary schooling, Descartes attended the University of Poitiers, receiving a law degree in 1616. He then embarked on a myriad of journeys, joining two different militaries (one in the Netherlands, the other in Bavaria) and generally travelling around Europe until 1628, when he settled in the Netherlands. It was here that he began to delve deeply into his ideas of science, mathematics, and philosophy.

In 1637, at the urging of his friends, Descartes published *Discourse on the Method for Conducting One's Reason Well and Seeking the Truth in the Sciences*. The book had three appendices: *La Dioptrique*, a work on optics; *Les Météores*, which pertained to meteorology; and *La Géométrie*, a work on mathematics. It was in this appendix that he proposed a geometric way of representing many different algebraic expressions and equations. It is this system of representation that almost all mathematical textbooks use today.

These publications (along with several others) brought much fame to Descartes. So renowned was his reputation that late in 1649, Queen Christina of Sweden asked Descartes

> to come to Sweden to tutor her. However, she wished to do her studies at 5:00 in the morning; Descartes had to break his lifelong habit of sleeping in late. A few months later, in February 1650, Descartes died of pneumonia.

Graphing Linear Inequalities

Previously we learned to solve inequalities with only one variable. We will now learn about inequalities containing two variables that can be written in one of the following forms: $Ax + By \geq C, Ax + By > C, Ax + By \leq C$, and $Ax + By < C$ where A and B are not both zero. We will look at **linear inequalities in two variables,** which are very similar to linear equations in two variables.

Like linear equations, linear inequalities in two variables have many solutions. Any ordered pair (x, y) that makes an inequality true when we substitute in the values is a **solution to a linear inequality**.

Example 5 **Determining Solutions to an Inequality**

Determine whether each ordered pair is a solution to the inequality $y > x + 4$:

1. $(0, 0)$
2. $(1, 6)$
3. $(2, 6)$
4. $(-5, -15)$
5. $(-8, 12)$

Solution

1.

$$(0,0) \quad y > x + 4$$

Substitute 0 for x and 0 for y $\quad 0 \overset{?}{>} 0 + 4$

Simplify. $\quad 0 \not> 4$

$(0, 0)$ is not a solution to $y > x + 4$.

2.

$$(1,6) \quad y > x + 4$$

Substitute 1 for x and 6 for y $\quad 6 \overset{?}{>} 1 + 4$

Simplify. $\quad 6 > 5$

$(1, 6)$ is a solution to $y > x + 4$.

3.

$$(2,6) \quad y > x + 4$$

Substitute 2 for x and 6 for y $\quad 6 \overset{?}{>} 2 + 4$

Simplify. $\quad 6 \not> 6$

$(2, 6)$ is not a solution to $y > x + 4$.

4.

$$(-5,-15)\quad y > x + 4$$

Substitute -5 for x and -15 for y: $-15 \overset{?}{>} -5 + 4$

Simplify. $-15 \not> -1$

So, $(-5,-15)$ is not a solution to $y > x + 4$.

5.

$$(-8,12)\quad y > x + 4$$

Substitute -8 for x and 12 for y: $12 \overset{?}{>} -8 + 4$

Simplify. $12 > -4$

So, $(-8,12)$ is a solution to $y > x + 4$.

Let us think about $x > 3$. The point $x = 3$ separated that number line into two parts. On one side of 3 are all the numbers less than 3. On the other side of 3 all the numbers are greater than 3.

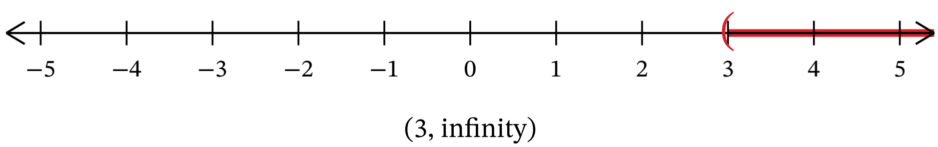


Figure 29: Solution to $x > 3$ on a Number Line

Similarly, the line $y = x + 4$ separates the plane into two regions. On one side of the line are points with $y < x + 4$. On the other side of the line are the points with $y > x + 4$. We call the line $y = x + 4$ a **boundary line**.

For an inequality in one variable, the endpoint is shown with a parenthesis or a bracket depending on whether or not a is included in the solution:

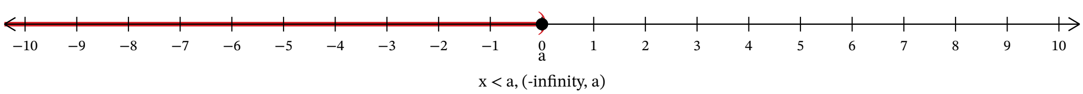


Figure 30: Endpoint with Parenthesis

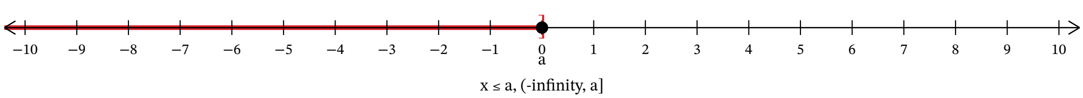


Figure 31: Endpoint with Bracket

Similarly, for an inequality in two variables, the boundary line is shown with a solid or dashed line to show whether or not it the line is included in the solution.

$Ax + By < C$	$Ax + By \leq C$
$Ax + By > C$	$Ax + By \geq C$
Boundary line is $Ax + By = C$	Boundary line is $Ax + By = C$
Boundary line is not included in solution.	Boundary line is included in solution.
Boundary line is dashed.	Boundary line is solid.

Now, let us take a look at what we found in Example 5.43. We will start by graphing the line $y = x + 4$, and then we will plot the five points we tested, as graphed in . We found that some of the points were solutions to the inequality $y > x + 4$ and some were not. Which of the points we plotted are solutions to the inequality $y > x + 4$? The points $(1, 6)$ and $(-8, 12)$ are solutions to the inequality $y > x + 4$. Notice that they are both on the same side of the boundary line $y = x + 4$. The two points $(0, 0)$ and $(-5, -15)$ are on the other side of the boundary line $y = x + 4$, and they are not solutions to the inequality $y > x + 4$. For those two points, $y < x + 4$. What about the point $(2, 6)$? Because $6 = 2 + 4$, the point is a solution to the equation $y = x + 4$, but not a solution to the inequality $y > x + 4$. So, the point $(2, 6)$ is on the boundary line.

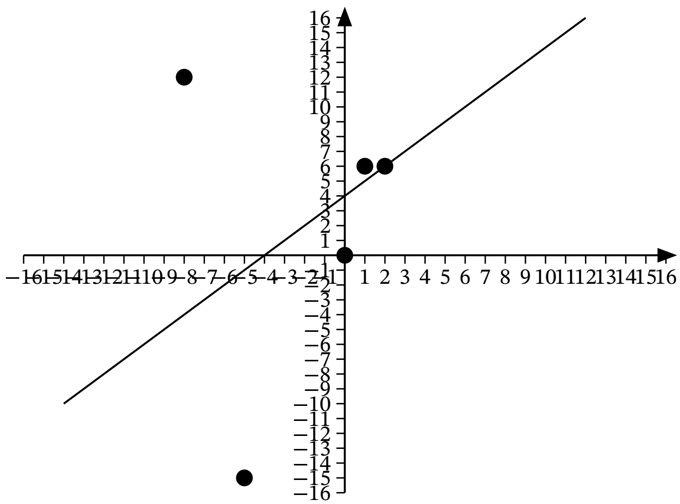


Figure 32: Graph of $y = x + 4$

Let us take another point above the boundary line and test whether or not it is a solution to the inequality $y > x + 4$. The point $(0, 10)$ clearly looks to be above the boundary line, doesn't it? Is it a solution to the inequality?

$$\begin{aligned} y &> x + 4 \\ 10 &\overset{?}{>} 0 + 4 \\ 10 &> \quad 4 \checkmark \end{aligned}$$

Yes, $(0, 10)$ is a solution to $y > x + 4$. Any point you choose above the boundary line is a solution to the inequality $y > x + 4$. All points above the boundary line are solutions. Similarly, all points below the boundary line, the side with $(0, 0)$ and $(-5, -15)$, are not solutions to $y > x + 4$, as shown.

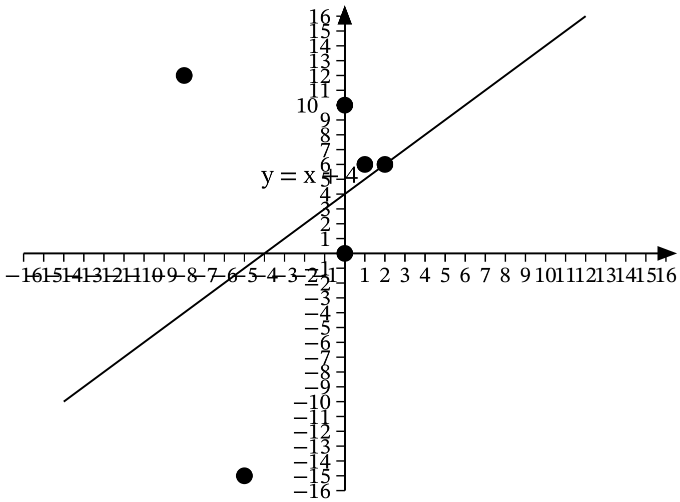


Figure 33: Graph of $y = x + 4$, with $y > x + 4$ Above the Boundary Line and $y < x + 4$ Below the Boundary Line

The graph of the inequality $y > x + 4$ is shown. The line $y = x + 4$ divides the plane into two regions. The shaded side shows the solutions to the inequality $y > x + 4$. The points on the boundary line, those where $y = x + 4$, are not solutions to the inequality $y > x + 4$, so the line itself is not part of the solution. We show that by making the boundary line dashed, not solid.

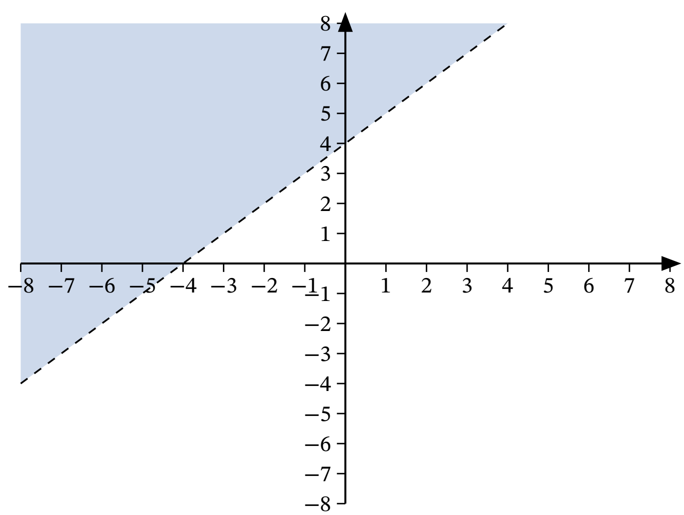
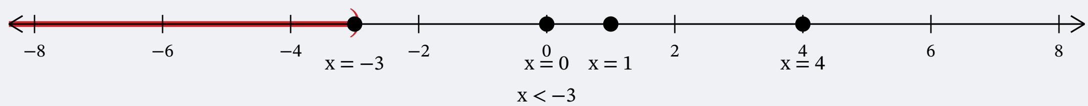


Figure 34: Graph of $y > x + 4$

Example 6 **Writing a Linear Inequality Shown by a Graph**

The boundary line shown in this graph is $y = 2x - 1$. Write the inequality shown.

Shaded Region to the Left of a Line



Solution

The line $y = 2x - 1$ is the boundary line. On one side of the line are the points with $y > 2x - 1$ and on the other side of the line are the points with $y < 2x - 1$. Let us test the point $(0, 0)$ and see which inequality describes its position relative to the boundary line. At $(0, 0)$, which inequality is true: $y > 2x - 1$ or $y < 2x - 1$?

$0 > 2(0) - 1$	$0 < 2(0) - 1$
$0 > 0 - 1$	$0 < 0 - 1$
$0 > -1$	$0 < -1$
True	False

Since $y > 2x - 1$ is true, the side of the line with $(0, 0)$, is the solution. The shaded region shows the solution of the inequality $y > 2x - 1$. Since the boundary line is graphed with a dashed line, the inequality does not include the equal sign. The graph shows the inequality $y > 2x - 1$.

We could use an y point as a test point, provided it is not on the line. Why did we choose $(0, 0)$? Because it is the easiest to evaluate. You may want to pick a point on the other side of the boundary line and check that $y < 2x - 1$.

Example 7 **Graphing a Linear Inequality**

Graph the linear inequality $y \geq \frac{3}{4}x - 2$.

Solution

Step 1. Identify and graph the boundary line . If the inequality is $\leq$ or $\geq$, the boundary line is solid. If the inequality is $<$ or $>$, the boundary line is dashed.	Replace the inequality sign with an equal sign to find the boundary line. Graph the boundary line $y = \frac{3}{4}x - 2$. The inequality sign is $\geq$, so we draw a solid line.	
Step 2. Test a point that is not on the boundary line. Is it a solution of the inequality?	We'll test $(0, 0)$. Is it a solution of the inequality?	At $(0, 0)$, is $y \geq \frac{3}{4}x - 2$? $0 \overset{?}{\geq} \frac{3}{4}(0) - 2$ $0 \geq -2$ So, $(0, 0)$ is a solution.
Step 3. Shade in one side of the boundary line . If the test point is a solution, shade in the side that includes the point. If the test point is not a solution, shade in the opposite side.	The test point $(0, 0)$ is a solution to $y \geq \frac{3}{4}x - 2$. So we shade in that side.	All points in the shaded region and on the boundary line represent the solution to $y \geq \frac{3}{4}x - 2$.

VIDEO

Graphing Linear Inequalities in Two Variables

Solving Applications Using Linear Inequalities in Two Variables

Many fields use linear inequalities to model a problem. While our examples may be about simple situations, they give us an opportunity to build our skills and to get a feel for how they might be used.

Example 8 **Working Multiple Jobs**

Hilaria works two part time jobs to earn enough money to meet her obligations of at least \$240 a week. Her job in food service pays \$10 an hour and her tutoring job on campus pays \$15 an hour. How many hours does Hilaria need to work at each job to earn at least \$240?

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. Write an inequality that would model this situation.
2. Graph the inequality.
3. Find three ordered pairs (x,y) that would be solutions to the inequality. Then, explain what that means for Hilaria.

Solution

1. Let x be the number of hours she works at the job in food service and let y be the number of hours she works tutoring. She earns \$10 per hour at the job in food service and \$15 an hour tutoring. At each job, the number of hours multiplied by the hourly wage will give the amount earned at that job.

$$\underbrace{\text{Amount earned at the food service job}}_{10x}\ \underbrace{\text{plus}}_{+}\ \underbrace{\text{the amount earned tutoring}}_{15y}\ \underbrace{\text{is at least}}_{\geq 240}$$

2. Graph the inequality: **Step 1:** Graph the boundary line $10x + 15y = 240$

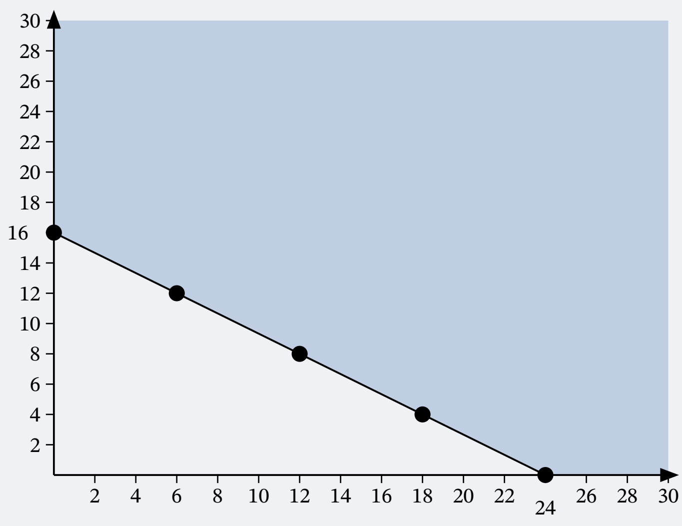
Create a table of values

x	y
0	$10(0) + 15y = 240 -> y = 16$
6	$10(6) + 15y = 240 -> y = 12$
12	$10(12) + 15y = 240 -> y = 8$

Step 2: Pick a test point. Let us pick $(0, 0)$ again:

$10(0) + 15(0) \geq 240$?

$0 \geq 240$ is false and not a solution so the shading happens on the other side of the boundary line .



1. From the graph, we see that the ordered pairs $(15, 10)$, $(0, 16)$, $(24, 0)$ represent three of infinitely many solutions. Check the values in the inequality.

$$(15, 10)$$
$$10x + 15y \geq 240$$
$$10(15) + 15(10) \overset{?}{\geq} 240$$
$$300 \geq 240 \text{ True}$$
$$(0, 16)$$
$$10x + 15y \geq 240$$
$$10(0) + 15(16) \overset{?}{\geq} 240$$
$$240 \geq 240 \text{ True}$$
$$(24, 0)$$
$$10x + 15y \geq 240$$
$$10(24) + 15(0) \overset{?}{\geq} 240$$
$$240 \geq 240 \text{ True}$$

For Hilaria, it means that to earn at least \$240, she can work 15 hours tutoring and 10 hours at her food service job, earn all her money tutoring for 16 hours, or earn all her money while working 24 hours at the job in food service.

Key Terms

- ordered pair
- origin
- points on the axes
- linear equation in two variables
- standards form of a linear equation
- solution
- linear inequality in two variables
- solution to a linear inequality
- boundary line

Key Concepts

- Linear equations can be represented graphically on a rectangular coordinate system.
- Solving linear equations in two variables means finding the point where two lines intersect. There are three possibilities: The lines intersect at exactly one point; the lines do not intersect (they are parallel); or the lines intersect everywhere (they are the same line).
- Solving linear inequalities in two variables means finding a region of possible answers. Every point in this region will make both inequalities true statements.
- Plotting points is a standard way to help graph linear equations and linear inequalities.

Videos

- Graphing Linear Inequalities in Two Variables

5.6 Quadratic Equations In One Variable with Applications

Figure 37: The Gateway Arch in St. Louis, Missouri

Learning Objectives

After completing this section, you should be able to:

1. Multiply binomials.
2. Factor trinomials.
3. Solve quadratic equations by graphing.
4. Solve quadratic equations by factoring.
5. Solve quadratic equations using square root method.
6. Solve quadratic equations using the quadratic formula.
7. Solve real world applications modeled by quadratic equations.

   In this section, we will discuss quadratic equations. There are several real-world scenarios that can be represented by the graph of a quadratic equation. Think of the Gateway Arch in St. Louis, Missouri. Both ends of the arch are 630 feet apart and the arch is 630 feet tall. You can plot these points on a coordinate system and create a parabola to graph the quadratic equation.

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a term is a constant, or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and x^m is a positive whole number, it is called a **monomial**. Some examples of monomial are 8, $-2x^2$, $4y^3$, and $11z$.

A monomial or two or more monomials combined by addition or subtraction is a **polynomial**. Some examples include: $b + 11$, $4y^2 - 7y + 2$, and $4x^4 + x^3 + 8x^2 - 9x + 1$. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term (examples: 14, $8y^2$, $-9x^3y^5$, and -13). A **binomial** has exactly two terms (examples:

$a + 7, 4b - 5, y^2 - 16$, and $3x^3 - 9x^2$), and a trinomial has exactly three terms (examples: $x^2 - 7x + 12, 9y^2 + 2y - 8, 6m^4 - m^3 + 8m$, and $x^4 + 3x^2 - 1$).

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words monomial, binomial, and trinomial when referring to these special polynomials and just call all the rest polynomials.

Multiply Binomials

Recall multiplying algebraic expressions from Algebraic Expressions. In this section, we will continue that work and multiply binomials as well. We can use an area model to do multiplication.

Example 1 **Multiply Binomials**

Multiply $(x + 2)(x + 3)$.

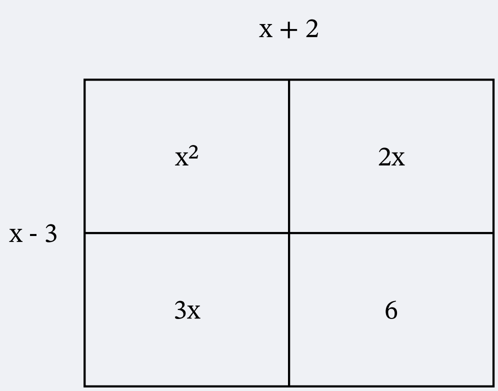
Solution

Step 1: Use the distributive property:

$$(x)(x) + (x)(3) + (2)(x) + (2)(3)$$

In the area model multiply each term on the side by each term on the top (think of it as a multiplication table).

Area Diagram



Step 2: After we multiply, we get the following equation:

$$x^2 + 3x + 2x + 6$$

Step 3: Combine the like terms to arrive at:

$$x^2 + 5x + 6$$

Example 2 **Multiplying More Complex Binomials**

Multiply $(2x + 7)(3x - 5)$.

Solution

Step 1: Use the Distributive Property:

$$(2x)(3x) - (2x)(5) + (7)(3x) - (7)(5)$$

Area Diagram

	2x	7
3x	$6x^2$	21x
−5	−10x	−35

Step 2: After multiplying, get the following equation:

$$6x^2 - 10x + 21x - 35$$

Step 3: Combine the like terms to arrive at:

$$6x^2 + 11x - 35$$

WHO KNEW?

They Are Teaching Multiplication of Binomials in Elementary School

Manipulatives are often used in elementary school for students to experience a hands-on way to experience the mathematics they are learning. Base Ten Blocks, or Dienes Blocks, are often used to introduce place value and the operation of whole numbers. When multiplying two-digit numbers, students can make an array to visualize the Distributive Property. shows the value of each Base Ten Block and shows how to multiply 17 and 23 using an area model and Base Ten Blocks. You can see how this helps students visualize the multiplication using the Distributive Property. Consider how $(10 + 7)(20 + 3)$ can easily extend to $(10 + x)(20 + y)$ in algebra!

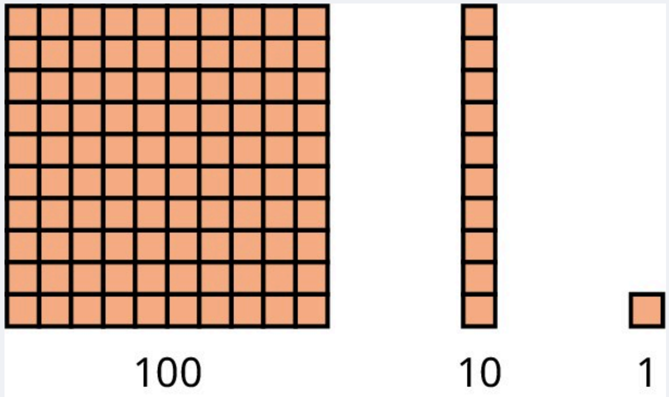


Figure 40: The Value of Each Base Ten Block

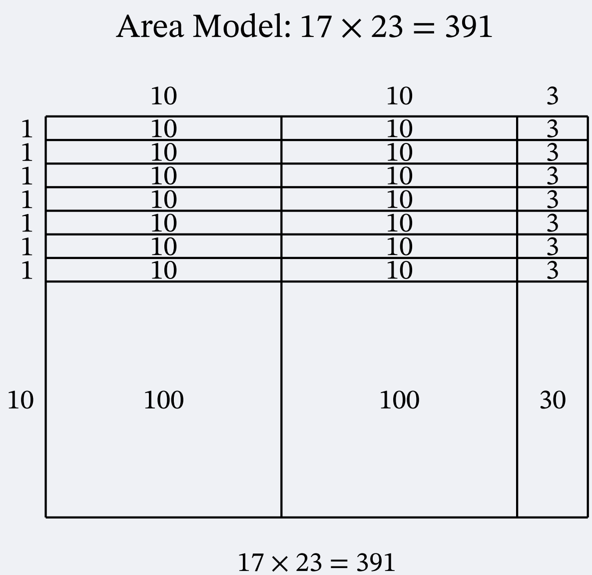


Figure 41: How to Multiply 17 and 23 Using an Area Model and Base Ten Blocks

Factoring Trinomials

We've just covered how to multiply binomials. Now you will need to "undo" this multiplication—to start with the product and end up with the factors. Let us review an example of multiplying binomials to refresh your memory.

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

To factor the trinomial means to start with the product, $x^2 + 5x + 6$, and end with the factors, $(x + 2)(x + 3)$. You need to think about where each of the terms in the trinomial came from. The first term came from multiplying the first term in each binomial. So, to get x^2 in the product, each binomial must start with an x.

$$x^2 + 5x + 6$$
$$(x)(x)$$

The last term in the trinomial came from multiplying the last term in each binomial. So, the last terms must multiply to 6. What two numbers multiply to 6? The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use? Consider the middle term. It came from adding the outer and inner terms. So the numbers that must have a product of 6 will need a sum of 5.

We'll test both possibilities and summarize the results in the following table, which will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of Factors
1, 6	$1 + 6 = 7$
2, 3	$2 + 3 = 5$

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. We have the factors of $x^2 + 5x + 6$. They are $(x + 2)(x + 3)$.

$$x^2 + 5x + 6 \text{ product}$$
$$(x + 2)(x + 3) \text{ factors}$$

You can check if the factors are correct by multiplying. Looking back, we started with $x^2 + 5x + 6$, which is of the form $x^2 + bx + c$, where $b = 5$ and $c = 6$. We factored it into two binomials of the form $(x + m)$ and $(x + n)$.

$$x^2 + 5x + 6x^2 + bx + c$$
$$(x + 2)(x + 3)(x + m)(x + n)$$

To get the correct factors, we found two number m and n whose product is c and sum is b. With the area model , start with an empty box and then put in the x^2 term and c.

x^2	
	6

Continue by putting in two terms that add up to $5x$: $2x$ and $3x$:

Area diagram

x^2	$2x$
$3x$	6

Then you find the terms of the binomials on the top and side :

	x	2
x	x^2	$2x$
3	$3x$	6

Example 3 **Factoring Trinomials**

Factor $x^2 + 7x + 12$.

Solution

The numbers that must have a product of 12 will need a sum of 7. We will summarize the results in a table below.

Factors of 12	Sum of Factors
1, 12	$1 + 12 = 13$
2, 6	$2 + 6 = 8$
3, 4	$3 + 4 = 7$

We see that 3 and 4 are the numbers that multiply to 12 and add to 7. The factors of $x^2 + 7x + 12$ are $(x + 3)(x + 4)$.

Example 4 **Factoring More Complex Trinomials**

Factor $x^2 - 11x + 28$.

Solution

The numbers that must have a product of 28 will need a sum of -11. We will summarize the results in a table.

Factors of 28	Sum of Factors
1, 28	$1 + 28 = 29$
2, 14	$2 + 14 = 16$
4, 7	$4 + 7 = 11$

We see that 4 and 7 are the numbers that multiply to 28 and add to 11. But we needed -11, so we will need to use -4 and -7 because $(-4)(-7) = 28$ and $(-4) + (-7) = -11$. The factors of $x^2 - 11x + 28$ are $(x - 4)(x - 7)$.

VIDEO

Factoring with the Box Method (Area Model)

Solving Quadratic Equations by Graphing

We have already solved and graphed linear equations in Graphing Linear Equations and Inequalities, equations of the form $Ax + By = C$. In linear equations, the variables have no exponents. **Quadratic equations** are equations in which the variable is squared. The following are some examples of quadratic equations:

$$x^2 + 5x + 6 = 03y^2 + 4y = 1064u^2 - 81 = 0n(n + 1) = 42$$

The last equation does not appear to have the variable squared, but when we simplify the expression on the left, we will get $n^2 + n$. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are real numbers, with $a \neq 0$. Remember that a solution of an equation is a value of a variable that makes a true statement when substituted into the equation. The solutions of quadratic equations are the values of the variables that make the quadratic equation $ax^2 + bx + c = 0$ true.

To solve quadratic equations, we need methods different than the ones we used in solving linear equations. We will start by solving a quadratic equation from its graph. Just like we started graphing linear equations by plotting points, we will do the same for quadratic equations. Let us look first at graphing the quadratic equation $y = x^2$. We will choose integer values of x between -2 and 2 and find their y values, as shown in the table below.

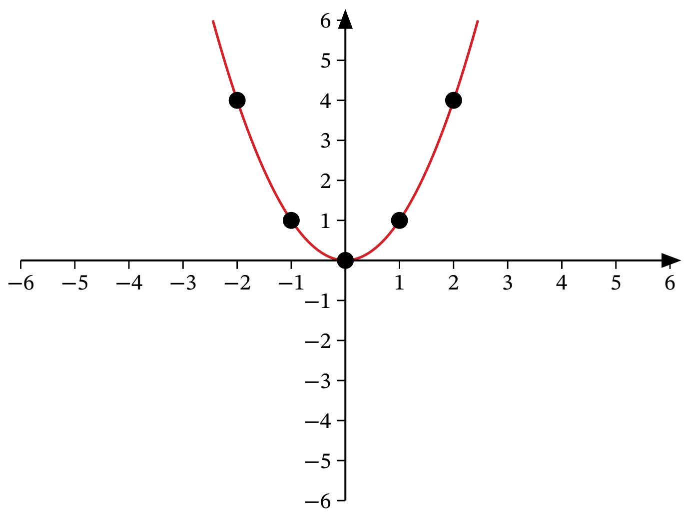
$y = x^2$	
$\boldsymbol{x}$	$\boldsymbol{y}$
0	0
1	1
−1	1
2	4
−2	4

Notice when we let $x = 1$ and $x = -1$, we got the same value for y.

$$y = x^2 y = 1^2 \qquad y = 1$$

$$y = x^2 y = (-1)^2 y = 1$$

The same thing happened when we let $x = 2$ and $x = -2$. Now, we will plot the points to show the graph of $y = x^2$.



The graph is not a line. This figure is called a parabola. Every quadratic equation has a graph that looks like this. When $y = 0$ the solution to the quadratic $y = x^2$ is 0 because $x^2 = 0$ at $x = 0$.

Example 5 **Graphing a Quadratic Equation**

Graph $y = x^2 - 1$ and list the solutions to the quadratic equation.

Solution

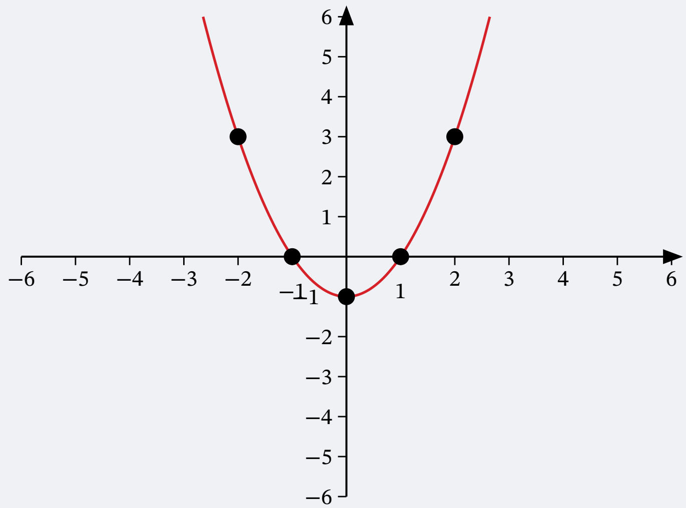
We will graph the equation by plotting points.

Step 1: Choose integer values for x, substitute them into the equation, and solve for y.

Step 2: Record the values of the ordered pairs in the chart.

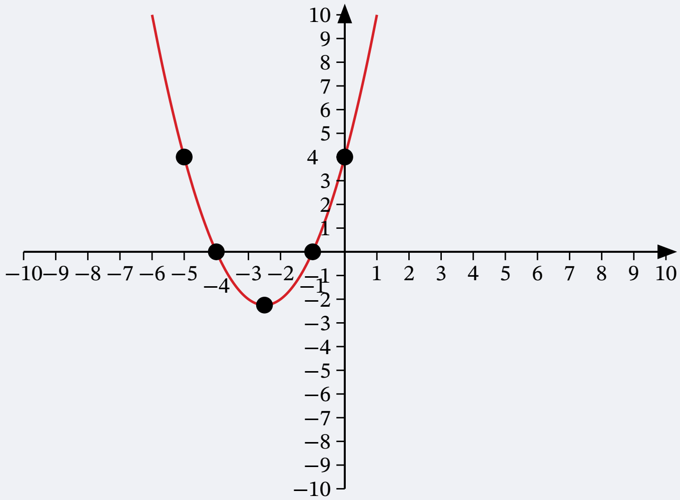
$y = x^2 - 1$	
x	y
0	−1
1	0
−1	0
2	3
−2	3

Step 3: Plot the points and then connect them with a smooth curve. The result will be the graph of the equation $y = x^2 - 1$. The solutions are $x = 1$ and $x = -1$.



Example 6 **Solving a Quadratic Equation From Its Graph**

Find the solutions of $y = x^2 + 5x + 4$ from its graph .



Solution

The solutions of a quadratic equations are the values of x that make the equation a true statement when set equal to zero (i.e. when $y = 0$). $x^2 + 5x + 4 = 0$ at $x = -4$ and $x = -1$.

Solving Quadratic Equations by Factoring

Another way of solving quadratic equations is by factoring. We will use the **Zero Product Property** that says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Example 7 **Solving a Quadratic Equation by Factoring**

Solve $(x + 1)(x - 4) = 0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0✓$
		$x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0✓$

Example 8 **Solve Another Quadratic Equation by Factoring**

Solve $x^2+2x-8=0$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	The equation is already in standard form.	$x^2 + 2x - 8 = 0$
Step 2. Factor the quadratic expression.	$x^2 + 2x - 8$ Factor $(x+4)(x-2)$	$(x+4)(x-2) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x + 4 = 0 \quad \text{or} \quad x - 2 = 0$
Step 4. Solve the linear equations.	We have two linear equations.	$x = -4 \quad \text{or} \quad x = 2$
Step 5. Check.	Substitute each solution separately into the original equation.	$\begin{aligned} x^2 + 2x - 8 &= 0 \\ x &= -4 \\ (-4)^2 - 2(-4) - 8 &\stackrel{?}{=} 0 \\ 16 + (-8) - 8 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$
		$\begin{aligned} x^2 + 2x - 8 &= 0 \\ x &= 2 \\ 2^2 - 2(2) - 8 &\stackrel{?}{=} 0 \\ 4 + 4 - 8 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$

VIDEO

Solving Quadratics with the Zero Property

CHECKPOINT

Be careful to write the quadratic equation in standard form first. The equation must be set equal to zero in order for you to use the Zero Product Property! Often students start in Step 2 resulting in an incorrect solution. For example, $x^2 + 2x - 15 = -7$ cannot be factored to $(x-3)(x+5) = -7$ and then solved by setting each factor equal to -7.

The correct way to solve this quadratic equation is to set it equal to zero FIRST: $x^2 + 2x - 15 + 7 = -7 + 7$ which becomes $x^2 + 2x - 8 = 0$, then continue to factor. See the table below for the correct way to apply the Zero Product Property.

	$x^2 + 2x - 15 = -7$	$x^2 + 2x - 15 = -7$		
Step 1	Skipped	$x^2 + 2x - 15 + 7 = -7 + 7$ $x^2 + 2x - 8 = 0$		
Step 2	$(x-3)(x+5) = -7$	$(x-2)(x+4) = 0$		
Step 3	$x - 3 = -7$	$x + 5 = -7$	$x - 2 = 0$	$x + 4 = 0$
Step 4	$x = -4$	$x = -12$	$x = 2$	$x = -4$
Step 5	$(-4)^2 + 2(-4) - 15$ $= 16 - 8 - 15$ $= -7$	$(-12)^2 + 2(-12) - 15$ $= 144 - 24 - 1 = 105$ $\neq -7$	$(2)^2 + 2(2) - 8$ $= 4 + 4 - 8$ $= 0$	$(-4)^2 + 2(-4) - 18$ $= 16 - 8 - 8$ $= 0$

Solving Quadratic Equations Using the Square Root Property

We just solved some quadratic equations by factoring. Let us use factoring to solve the quadratic equation $x^2 = 9$.

Step 1: Put the equation in standard form.	$x^2 - 9 = 0$	
Step 2: Factor the left side.	$(x+3)(x-3) = 0$	
Step 3: Use the Zero Product Property.	$x + 3 = 0$	$x - 3 = 0$
Step 4: Solve each equation.	$x = -3$	$x = 3$
Step 5: Combine the two solutions into ±	$x = \pm 3$	

The solution is read as "x is equal to positive or negative 3."

What happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring. These equations are all of the form $x^2 = k$. We define the square root of a number in this way: If $n^2 = m$, then n is a square root of m. This leads to the **Square Root Property**.

Example 9 **Using the Square Root Property to Solve a Quadratic Equation**

Solve using the square Root Property: $x^2 = 169$.

Solution

Step 1: Use the Square Root Property.	$x = \pm\sqrt{169}$	
Step 2: Simplify the radical.	$x = \pm 13$	
Step 3: Rewrite to show the two solutions.	$x = 13$	$x = -13$

Example 10 **Using the Square Root Property to Solve Another Quadratic Equation**

Solve using the Square Root Property: $144q^2 = 25$.

Solution

Step 1: Solve for q.

$$q^2 = \frac{25}{144}$$

Step 2: Use the Square Root Property.

$$q = \pm\sqrt{\frac{25}{144}}$$

Step 3: Simplify the radical.

$$q = \pm\frac{5}{12}$$

Step 4: Rewrite to show the two solutions.

$q = \frac{5}{12}, q = -\frac{5}{12}$

Solving Quadratic Equations Using the Quadratic Formula

This last method we will look at for solving quadratic equations is the **quadratic formula**. This method works for all quadratic equations, even the quadratic equations we could not factor! To use the quadratic formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

Example 11 **Solving a Quadratic Equation Using the Quadratic Formula**

Solve using the quadratic formula: $x^2 - 6x + 5 = 0$.

Solution

$$x^2 - 6x + 5 = 0$$

This equation is in standard form.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 - 6x + 5 &= 0 \end{aligned}$$

Step 1: Identify the a, b, and c values.

$$a = 1, b = -6, c = 5$$

Step 2: Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 3: Substitute in the values of a, b, c.

$$x = \frac{(-6) \pm \sqrt{(-6)^2 - 4(1)(5)}}{2.1}$$

Step 4: Simplify.

$$x = \frac{6 \pm \sqrt{36 - 20}}{2}$$

$$x = \frac{6 \pm \sqrt{16}}{2}$$

$$x = \frac{6 \pm 4}{2}$$

Step 5: Rewrite to show two solutions.

$$x = \frac{6 + 4}{2}, x = \frac{6 - 4}{2}$$

Step 6: Simplify.

$$x = \frac{10}{2}, x = \frac{2}{2}$$

$$x = 5, x = 1$$

Step 7: Check.

$$\begin{array}{rll} x^2 - 6x + 5 = & 0x^2 - 6x + 5 & = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} & 01^2 - 6 \cdot 1 + 5 & \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} & 01 - 6 + 5 & \stackrel{?}{=} 0 \\ 0 = 0\checkmark 0 & & = 0\checkmark \end{array}$$

Example 12 **Solving Another Quadratic Equation Using the Quadratic Formula**

Solve using the quadratic formula: $2x^2 + 9x - 5 = 0$.

Solution

Step 1. Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$\boldsymbol{ax}^2 + \boldsymbol{bx} + \boldsymbol{c} = 0$ $2x^2 + 9x - 5 = 0$ $a = 2,\ b = 9,\ c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $\boldsymbol{a} = 2$, $\boldsymbol{b} = 9$, $\boldsymbol{c} = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4(2)(-5)}}{2 \cdot 2}$
Step 3. Simplify the fraction, and solve for x.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4} \quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4} \quad x = \frac{-20}{4}$ $x = \frac{1}{2} \quad x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0 ✓$
	Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0 ✓$

VIDEO

Solving Quadratics with the Quadratic Formula

Solving Real-World Applications Modeled by Quadratic Equations

There are problem solving strategies that will work well for applications that translate to quadratic equations. Here's a problem-solving strategy to solve word problems:

Step 1: Read the problem. Make sure all the words and ideas are understood.

Step 2: Identify what we are looking for.

Step 3: Name what we are looking for. Choose a variable to represent that quantity.

Step 4: Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5: Solve the equation using good algebra techniques.

Step 6: Check the answer in the problem and make sure it makes sense.

Step 7: Answer the question with a complete sentence.

Example 13 **Finding Consecutive Integers**

The product of two consecutive integers is 132. Find the integers.

Solution

Step 1: Read the problem.

Step 2: Identify what we are looking for.

We are looking for two consecutive integers.

Step 3: Name what we are looking for.

Let $n =$ the first integer

Let $n + 1 =$ the next consecutive integer.

Step 4: Translate into an equation. Restate the problem in a sentence.

$$n(n + 1) = 132$$

The product of the two consecutive integers is 132. The first integer times the next integer is 132.

Step 5: Solve the equation.

$$n^2 + n = 132$$

Bring all the terms to one side.

$$n^2 + n - 132 = 0$$

Factor the trinomial.

$$(n - 11)(n + 12) = 0$$

Use the zero product property.

$$n - 11 = 0 \text{ or } n + 12 = 0$$

Solve the equations.

$$n = 11, n = -12$$

There are two values for n that are solutions to this problem. So, there are two sets of consecutive integers that will work.

If the first integer is $n = 11$, then the next integer is 12. If the first integer is $n = -12$, then the next integer is -11.

Step 6: Check the answer.

The consecutive integers are 11, 12 and -11, -12. The product of 11 and 12 = 132 and the product of $-11(-12) = 132$. Both pairs of consecutive integers are solutions.

Step 7: Answer the question.

The consecutive integers are 11, 12, and -11, -12.

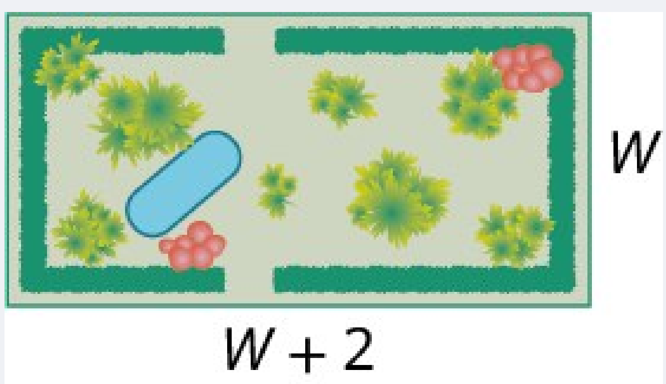
Were you surprised by the pair of negative integers that is one of the solutions? In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

Example 14 **Finding Length and Width of a Garden**

A rectangular garden has an area 15 square feet. The length of the garden is 2 feet more than the width. Find the length and width of the garden.

Solution

Step 1: Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation .



Step 2: Identify what you are looking for.

We are looking for the length and width.

Step 3: Name what you are looking for. The length is 2 feet more than width.

Let W = the width of the garden.

$W + 2$ = the length of the garden.

Step 4: Translate into an equation.

Restate the important information in a sentence.

The area of the rectangular garden is 15 square feet.

Use the formula for the area of a rectangle.

$$A = L \times W$$

Substitute in the variables.

$$15 = (W + 2)W$$

Step 5: Solve the equation. Distribute first.

$$15 = W^2 + 2W$$

Get zero on one side.

$$0 = W^2 + 2W - 15$$

Factor the trinomial.

$$0 = (W + 5)(W - 3)$$

Use the Zero Product Property.

$$0 = W + 5$$

$$0 = W - 3$$

Solve each equation.

$$-5 = W$$

$$3 = W$$

Since W is the width of the garden, it does not make sense for it to be negative. We eliminate that value for W.

$$W = 3$$

Width is 3 feet.

Find the value of the length.

$W + 2 =$ length.

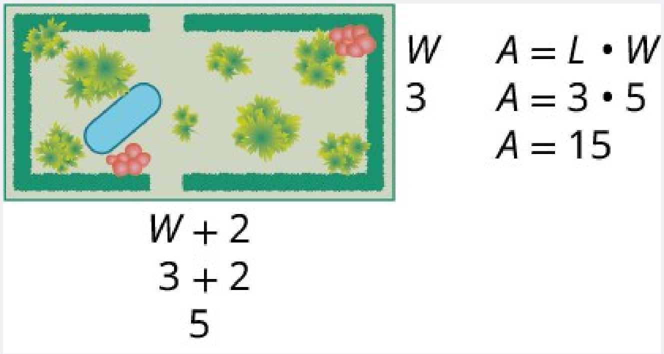
$$3 + 2$$

5

Length is 5 feet.

Step 6: Check the answer .

Does the answer make sense?



Yes, this makes sense.

Step 7: Answer the question.

The width of the garden is 3 feet and the length is 5 feet.

WORK IT OUT

Completing the Square

Recall the two methods used to solve quadratic equations of the form $ax^2 + bx + c$: by factoring and by using the quadratic formula. There are, however, many different methods for solving quadratic equations that were developed throughout history. Egyptian, Mesopotamian, Chinese, Indian, and Greek mathematicians all solved various types of quadratic equations, as did Arab mathematicians of the ninth through the twelfth centuries. It is one of these Arab mathematicians' methods that we wish to investigate with this activity.

Muhammad ibn Musa al-Khwarizmi was employed as a scholar at the House of Wisdom in Baghdad, located in present day Iraq. One of the many accomplishments of Al-Khwarizmi was his book on the topic of algebra. In that book, he asks, "What must be the square which, when increased by ten of its own roots, amounts to 39?" Al-Khwarizmi, like many Arab mathematicians of his time, was well versed in Euclid's Elements. Like Euclid, he viewed algebra very geometrically, and thus had a geometric approach to solving a problem like the one above. In his approach, he used a method which today we refer to as *completing the square*.

His description of the solution method for the above problem is: halve the number of roots, which in the present instance yields 5. This you multiply by itself; the product is 25. Add this to 39; the sum is 64. Now take the root of this which is 8, and subtract from it half the number of the roots, which is 5; the remainder is 3. This is the root of the square which you sought for. Thus the square is 9.

So, what does all of this mean? Al-Khwarizmi would start with a square of unknown length of side (we will label the side length x). See So, this square has area x^2	x
He would then proceed to halve the number of roots (i.e., there are 10 roots by which the square is increasing) to get 5; this he would add to the first square. See The area of the two new pieces added into the original square are both $5x$. At this point, we have $x^2 + 10x = 39$.	
Now Al-Khwarizmi needed to "complete the square" by adding into the drawing a small square. See This square has an area of 25.	

$x^2 + 10x + 25 = 39 + 25$, or $x^2 + 10x + 25 = 64$.

Notice that the completed square has side length $x + 5$, so the large square has area $(x + 5)^2$. (Notice algebraically that the left half of the equation $x^2 + 10x + 25 = 64$ factors to $(x + 5)^2 = 64$.) This means the area of large square equals 64. If $(x + 5)^2 = 64$, then $x + 5 = 8$ so x must be equal to 3 or -13 to make this true. Note that Al-Kwarimi would not have considered the possibility of a negative solution, since he approached the solution geometrically, and negative distances do not exist.

Key Terms

- monomial
- polynomial
- binomial
- trinomial
- quadratic equation
- Zero Product Property

Key Concepts

- A quadratic equation is an algebraic equation where the highest power (degree) of the equation is two.
- To solve a quadratic equation is to find the value(s) that when substituted in for the variables, will make the equation equal to zero.
- There can be two, one, or no solutions to any quadratic equation.
- There are several methods to solve a quadratic equation. These methods include factoring quadratic equations, graphic quadratic equations, using the square root method, and using the quadratic formula.

Videos

- Factoring with the Box Method (Area Model)
- Solving Quadratics with the Zero Property

- Solving Quadratics with the Quadratic Formula

5.7 Functions

Figure 50: A small group of elementary students learning from their teacher.

Learning Objectives

After completing this section, you should be able to:

1. Use function notation.
2. Determine if a relation is a function with different representations.
3. Apply the vertical line test.
4. Determine the domain and range of a function.

In this section, we will learn about relations and functions. As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number, and our birthday are matched to our name. There is a relationship between our name and each of those items. When your teacher gets their class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a relation.

(Student name, Student ID #)

The set of all the names of the students in the class is called the domain of the relation and the set of all student ID numbers paired with these students is the range of the relation. In general terms, a **relation** is any set of ordered pairs, (x, y). All the x-values in the ordered pairs together make up the **domain**. All the y-values in the ordered pairs together make up the range.

There are many situations similar to the student's name and student ID # where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation. A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each x-value is matched with only one y-value.

Let us look at the relation between your friends and their birthdays in . Every friend has a birthday, but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24 as their birthday and that June and Liz share August 2. Since each person has exactly one birthday, the relation is a function.

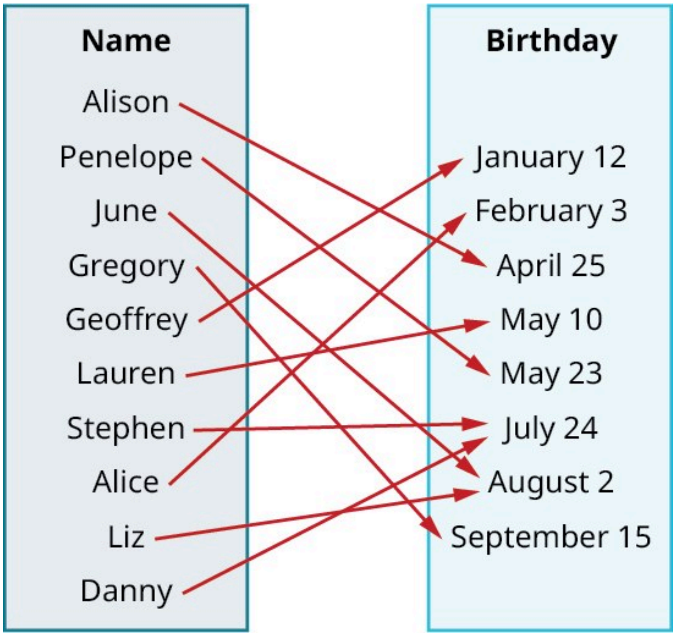


Figure 51: Birthday Mapping

Use Function Notation

It is very convenient to name a function; most often functions are named f, g, h, F, G, or H. In any function, for each x-value from the domain, we get a corresponding y-value in the range. In the function f, we write this range value y as $f\ (x)$. This notation $f\ (x)$ is called function notation and is read "f of x" or "the value of f at x." In this case the parentheses do not indicate multiplication.

We call x the independent variable as it can be any value in the domain. We call y the dependent variable as its value depends on x. Much like when you first encountered the variable x, function notation may be rather unsettling. But the more you use the notation, the more familiar you become with the notation, and the more comfortable you will be with it.

Let's review the equation $y = 4x - 5$. To find the value of y when $x = 2$, we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let $x = 2$.	$y = 4 \cdot 2 - 5$ $y = 3$

The value of the function at $x = 2$ is 3. We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let $x = 2$.	$f(2) = 4 \cdot 2 - 5$ $f(2) = 3$

The value of the function at $x = 2$ is 3. This process of finding the value of $f(x)$ for a given value of x is called **evaluating the function**.

Example 1 **Evaluating the Function**

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

1. $f(3)$
2. $f(-2)$
3. $f(a)$

Solution

1. To evaluate $f(3)$, substitute 3, for x.
   Simplify.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 1 \\ f(3) &= 2(3)^2 + 3 \cdot 3 - 1 \\ f(3) &= 2 \cdot 9 + 3 \cdot 3 - 1 \\ f(3) &= 18 + 9 - 1 \\ f(3) &= 26 \end{aligned}$$

2. To evaluate $f(-2)$, substitute -2 for x.
   Simplify.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 1 \\ f(-2) &= 2(-2)^2 + 3(-2) - 1 \\ f(-2) &= 2 \cdot 4 + (-6) - 1 \\ f(-2) &= 8 + (-6) - 1 \\ f(-2) &= 1 \end{aligned}$$

3. To evaluate $f(a)$, substitute a for x.
   Simplify.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 1 \\ f(a) &= 2(a)^2 + 3 \cdot a - 1 \\ f(a) &= 2a^2 + 3a - 1 \end{aligned}$$

Example 2 **Evaluating the Function in an Application**

The number of unread emails in Sylvia's inbox is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, N, and the time, t, measured in days. Find N (5). Explain what this result means.

Solution

Find N (5). Explain what this result means.

Substitute in $t = 5$.

Simplify.

$$\begin{aligned} N(5) &= 75 + 10 \cdot 5 \\ N(t) &= 75 + 10t \\ N(5) &= 75 + 50 \\ N(5) &= 125 \end{aligned}$$

If 5 is the number of days, $N(5)$ is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in Sylvia's inbox.

Determining If a Relation Is a Function with Different Representations

We can determine whether a relation is a function by identifying the input and the output values. If each input value leads to only one output value, classify the relation as a function. If any input value leads to two or more outputs, do not classify the relation as a function.

We will review three different representations of relations and determine if they are functions: ordered pairs, mapping, and equations.

Example 3 **Determining If a Relation Is a Function with a Set of Ordered Pairs**

Use the set of ordered pairs to determine whether the relation is a function.

1. $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$
2. $\{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$

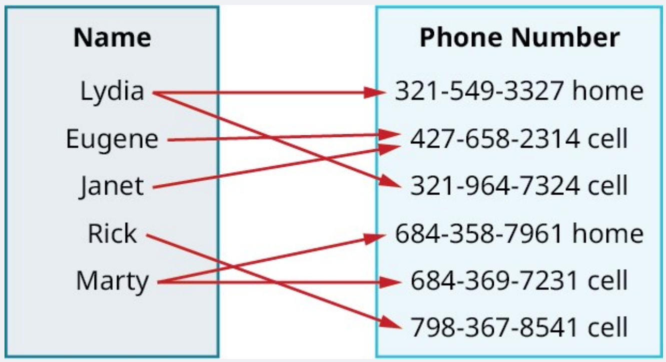
Solution

1. Each x-value is matched with only one y-value. This relation is a function.
2. The x-value 9 is matched with two y-values, both 3 and -3. This relation is not a function.

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range. Consider the example of the relation between your friends and their birthdays used in . In this particular example, the domain is the set of people's names, and the range is the set of their birthdays. This mapping was a function because everybody's name maps to exactly one birthday.

Example 4 **Determining If a Relation Is a Function with Mapping**

Use the mapping in to determine whether the relation is a function.



Solution

Both Lydia and Marty have two phone numbers. Each x-value is not matched with only one y-value. This relation is not a function.

In algebra functions will usually be represented by an equation. It is easiest to see if the equation is a function when it is solved for y. If each value of x results in only one value of y, then the equation defines a function.

Example 5 **Determining If a Relation Is a Function with an Equation**

Determine whether each equation is a function. Assume x is the independent variable.

1. $2x + y = 7$
2. $y = x^2 + 1$
3. $x + y^2 = 3$

Solution

1. $2x + y = 7$ For each value of x, we multiply it by -2 and then add 7 to get the y-value. For example, if $x = 3$:

$$\begin{aligned} y &= \quad -2x + 7 \\ y &= -2 \cdot 3 + 7 \\ y &= \qquad\quad 1 \end{aligned}$$

We have that when $x = 3$, then $y = 1$. It would work similarly for any value of x. Since each value of x, corresponds to only one value of y the equation defines a function.

1. $y = x^2 + 1$ For each value of x, we square it and then add 1 to get the y-value.

For example, if $x = 2$

$$\begin{aligned} y &= x^2 + 1 \\ y &= 2^2 + 1 \\ y &= \quad 5 \end{aligned}$$

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of x. Since each value of x corresponds to only one value of y, the equation defines a function.

1. $x + y^2 = 3$

   $x + y^2 = 3$

Isolate the y term.

$$y^2 = -x + 3$$

Let us substitute $x = 2$.

$$y^2 = -2 + 3$$

$$y^2 = 1$$

This gives us two values for y.

$$y = 1, y = -1$$

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of x. Since each value of x does not corresponds to only one value of y the equation does not define a function.

VIDEO

Relations and Functions

Applying the Vertical Line Test

We reviewed how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping, or an equation. We will now cover how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the x-values and y-values of the ordered pair are substituted into the equation. The graph of a linear equation is a straight line where every point on the line is a solution of the equation, and every solution of this equation is a point on this line. we can see that in the graph of the equation $y = 2x - 3$, for every x-value there is only one y-value, as shown in the accompanying table.

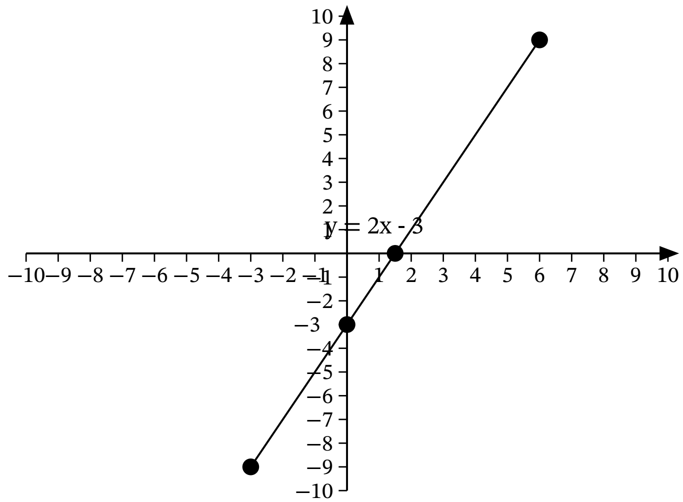
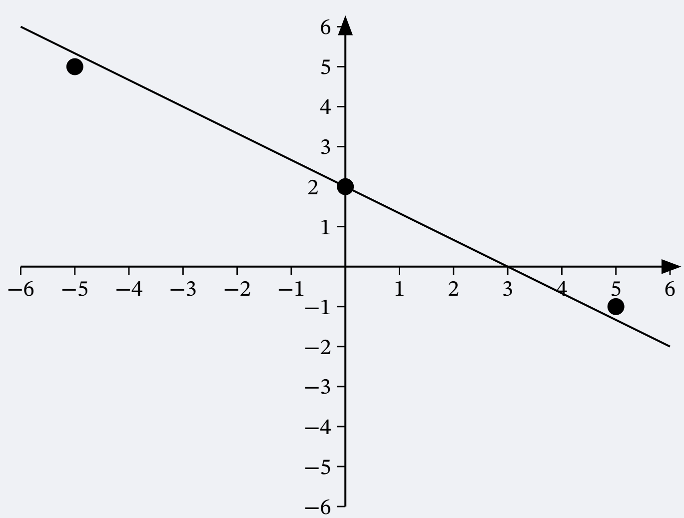


Figure 53: Graph of the Equation $y = 2x - 3$

A relation is a function if every element of the domain has exactly one value in the range. The relation defined by the equation $y = 2x - 3$ is a function. If we look at the graph, each vertical dashed line only intersects the solid line at one point. This makes sense as in a function, for every x-value there is only one y-value. If the vertical line hit the graph twice, the x-value would be mapped to two y-values, and so the graph would not represent a function. This leads us a graphical method of determining functions called the **vertical line test**, which states that a set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

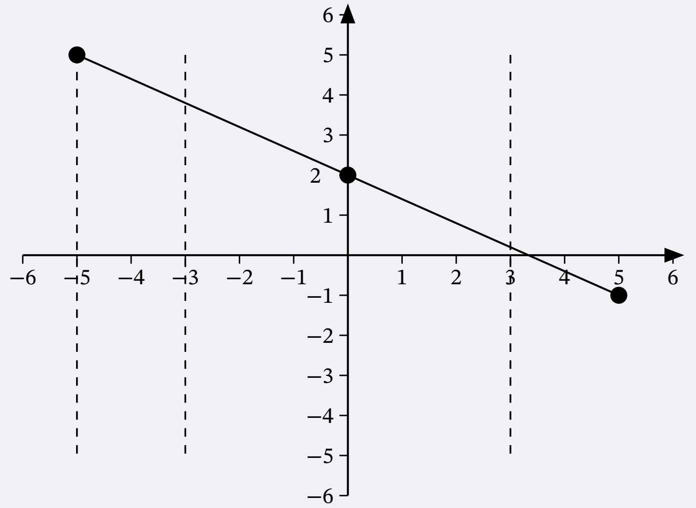
Example 6 **Applying the Vertical Line Test**

Determine whether the graph is the graph of a function applying the vertical line test.



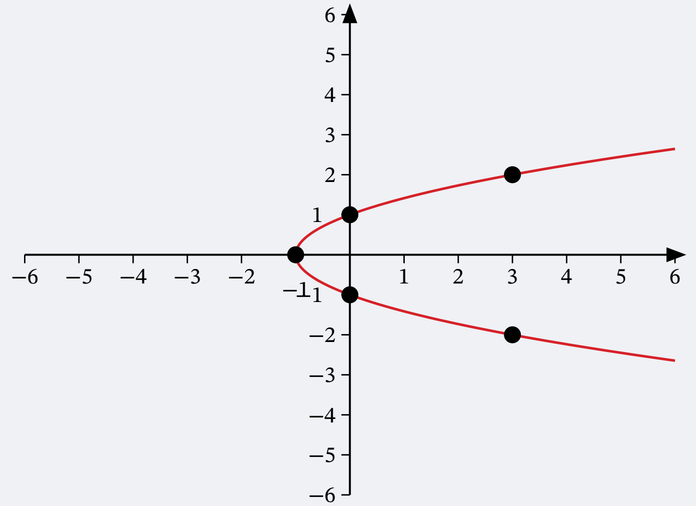
Solution

On the graph , only three vertical dashed lines are drawn. However, it can be determined that any vertical dashed line that is drawn will intersect the solid line at exactly one point. It is the graph of a function.



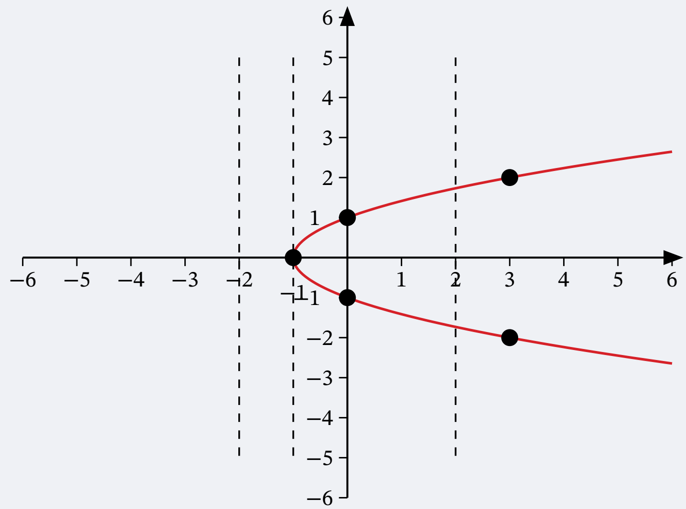
Example 7 **Applying the Vertical Line Test to a Parabola**

Determine whether the graph is the graph of a function .



Solution

does not represent a function since the vertical dashed lines shown on the graph below intersect the solid line at two points.



Determining the Domain and Range of a Function

For the function $y = f(x)$, x is the independent variable as it can be any value in the domain, and y is the dependent variable since its value depends on x. For the function $y = f(x)$, the values of x make up the domain and the values of y make up the range.

Example 8 **Finding the Domain and Range of Ordered Pairs**

For $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$:

1. Find the domain of the relation.
2. Find the range of the relation.

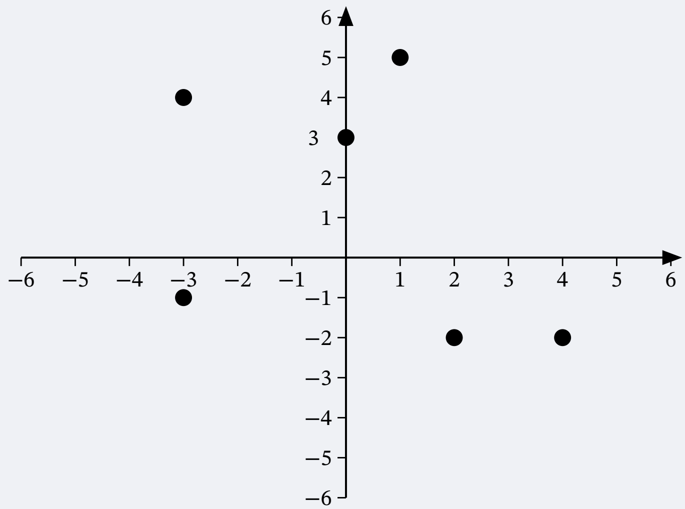
Solution

1. The domain is the set of all x-values of the relation: $\{1, 2, 3, 4, 5\}$
2. The range is the set of all y-values of the relation: $\{1, 4, 9, 16, 25\}$

Example 9 **Finding the Domain and Range on a Graph**

Use to:

1. List the ordered pairs of the relation.
2. Find the domain of the relation.
3. Find the range of the relation.



Solution

1. The ordered pairs of the relation are: $\{(1, 5), (-3, -1), (4, -2), (0, 3), (2, -2), (-3, 4)\}$.
2. The domain is the set of all x-values of the relation: $\{-3, 0, 1, 2, 4\}$. Notice that while -3 repeats, it is only listed once.
3. The range is the set of all y-values of the relation: $\{-2, -1, 3, 4, 5\}$. Notice that while -2 repeats, it is only listed once.

VIDEO

Domain and Range on Graphs

WHO KNEW?

Function and Function Notation

In 1673, Gottfried Leibniz, the German mathematician who co-invented calculus, seems to be the first person to use the word *function* in a mathematical sense, although his use of it does not exactly fit with the modern use and definition. The person who is credited with the modern definition of function is Swiss mathematician Johann Bernoulli, who wrote about it in a letter to Leibniz in 1698. Supposedly, Leibniz wrote Bernoulli back, approving of this use of the word. In 1734, the use of the notation $f(x)$ for a function was first used by Swiss mathematician Leonhard Euler (pronounced “Oiler”). Euler had a knack for inventing notation. He also introduced the notation e for the base of natural logs (1727), i for the square root of -1 (1777), $\sum$ for summation (1755), and many others. Euler also introduced many other ideas associated with functions. Euler defined exponential functions and defined logarithmic functions as their inverse; he also introduced the beta and gamma functions, and was the first person to consider the trigonometric identities (sine, cosine, etc.) as functions.

Key Terms

- relation
- domain
- function
- mapping
- vertical line test

Key Concepts

- A relation is any set of ordered pairs (x, y). All of the x-values of the set are the domain, and all of the y-values of the set are the range.
- A relation is a function if each x-value in the domain is assigned to exactly one element in the range. A y-value in the range can have more than one x-value assigned to it; but each x-value can only be assigned to one y-value.
- For the function $y = f(x)$, f is the name of the function, x is the domain value variable, and $y = f(x)$ is the range value variable.
- The vertical line test is a test that can be done on the graph of a relation to determine if it is a function.

Videos

- Relations and Functions
- Domain and Range on Graphs

5.8 Graphing Functions

Figure 59: The ski lifts and the mountain both have a slope.

Learning Objectives

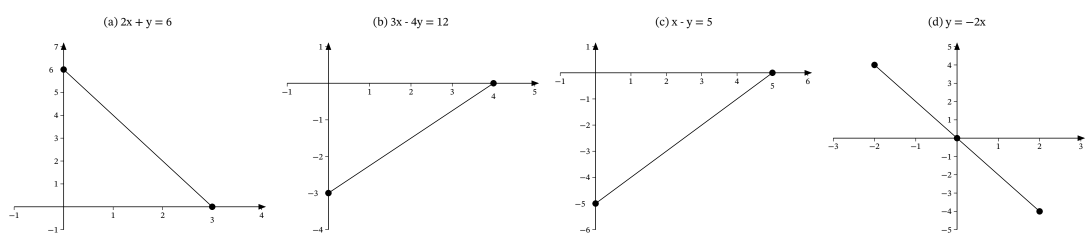
After completing this module, you should be able to:

1. Graph functions using intercepts.
2. Compute slope.
3. Graph functions using slope and y-intercept.
4. Graph horizontal and vertical lines.
5. Interpret graphs of functions.
6. Model applications using slope and y-intercept.

   In this section, we will expand our knowledge of graphing by graphing linear functions. There are many real-world scenarios that can be represented by graphs of linear functions. Imagine a chairlift going up at a ski resort. The journey a skier takes travelling up the chairlift could be represented as a linear function with a positive slope. The journey a skier takes down the slopes could be represented by a linear function with a negative slope.

Graphing Functions Using Intercepts

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points. At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the x-axis and the y-axis. These points are called the **intercepts** of a line. Let us review the graphs of the lines in .

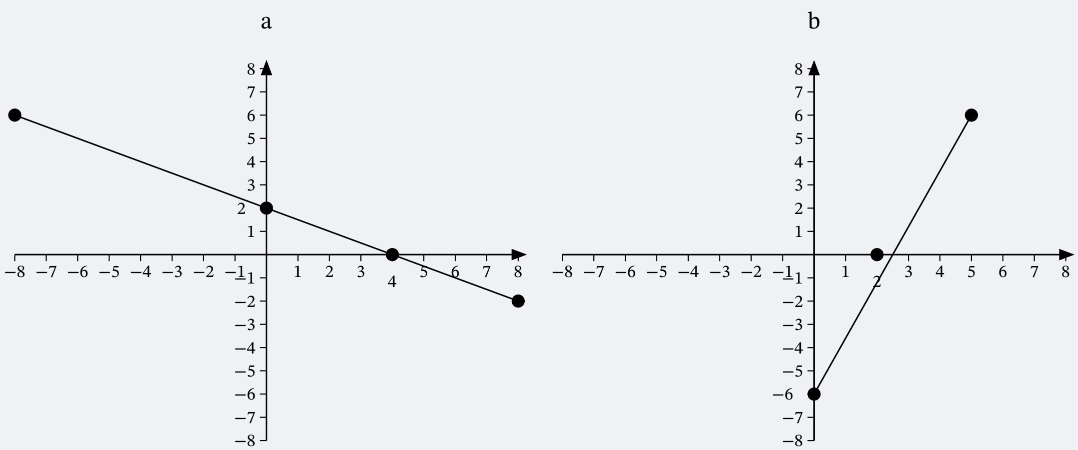


The table below lists where each of these lines crosses the x- and y-axis. Do you see a pattern? For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a, 0)$ and is called the x-intercept of the line. The x-intercept occurs when y is zero. In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the y-intercept of the line. The y-intercept occurs when x is zero.

Figure	**The line crosses the x -axis at:**	**Ordered Pair for this Point**	**The line crosses the y -axis at:**	**Ordered Pair for This Point**
Figure (a)	3	$(3, 0)$	6	$(0, 6)$
Figure (b)	4	$(4, 0)$	-3	$(0, -3)$
Figure (c)	5	$(5, 0)$	-5	$(0, -5)$
Figure (d)	0	$(0, 0)$	0	$(0, 0)$
General Figure	a	$(a, 0)$	b	$(0, b)$

Example 1 **Finding x- and y-Intercepts**

Find the x-intercept and y-intercept on the (a) and (b) graphs in .



Solution

In , the graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$. The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$. In , the graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$. The graph crosses the y-axis at the point $(0, -6)$. The y -intercept is $(0, -6)$.

Example 2 **Graphing a Function Using Intercepts**

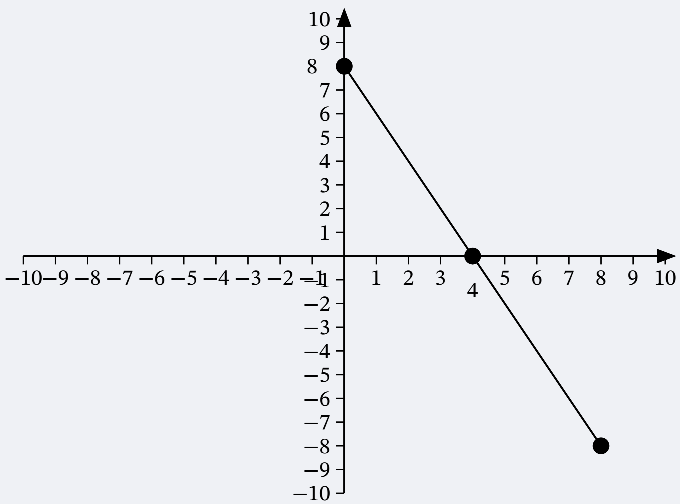
Find the intercepts of $2x + y = 8$. Then graph the function using the intercepts.

Solution

Let $y = 0$ to find the x-intercept, and let $x = 0$ to find the y-intercept.

	$2x + y = 8$		$2x + y = 8$
To find the x-intercept, let $y = 0$.	$2x + 0 = 8$	To find the y-intercept, let $x = 0$.	$2(0) + y = 8$
Simplify.	$2x = 8$ $x = 4$	Simplify.	$y = 8$
The x-intercept is:	$(4, 0)$	The y-intercept is:	$(0, 8)$

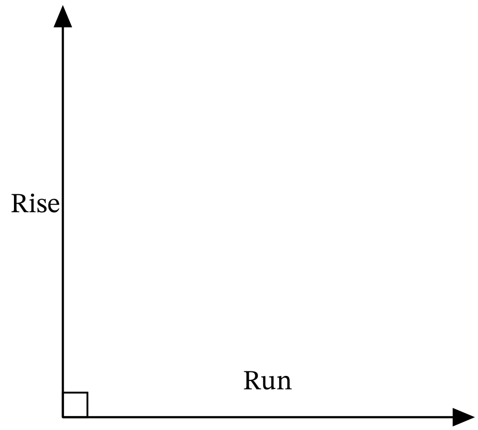
Plot the intercepts to get the graph in .



Computing Slope

When graphing linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. In mathematics, the measure of the steepness of a line is called the **slope** of the line. To find the slope of a line, we locate two points on the line whose coordinates are integers. Then we sketch a right triangle where the two points are vertices of the triangle and one side is horizontal and one side is vertical. Next, we measure or calculate the distance along the vertical and horizontal sides of the triangle. The vertical distance is called the **rise** and the horizontal distance is called the **run**.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The rise is the amount the vertical distance changes while the run measures the horizontal change, as shown in this illustration. Slope is a rate of change.



FORMULA

To calculate slope (m), use the formula

$$m = \frac{rise}{run}$$

where the rise measures the vertical change and the run measures the horizontal change.

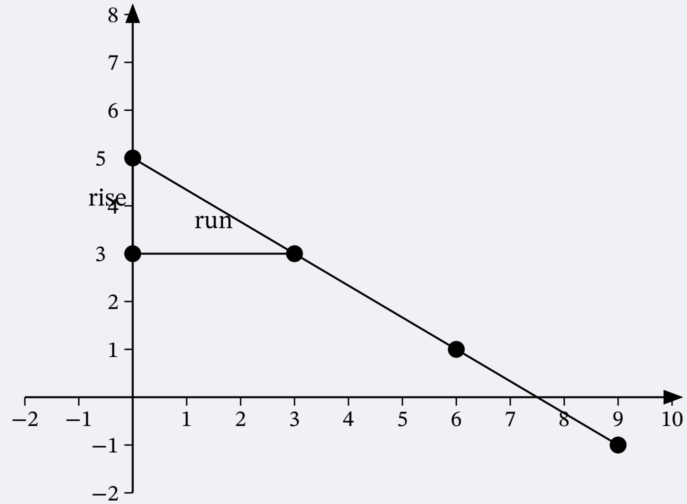
The concept of slope has many applications in the real world. In construction, the pitch of a roof, the slant of plumbing pipes, and the steepness of stairs are all applications of slope. As you ski or jog down a hill, you definitely experience slope.

Example 3 **Finding the Slope from a Graph**

Find the slope of the line shown.

Solution

Step 1: Locate two points on the graph whose coordinates are integers, such as $(0, 5)$ and $(3, 3)$. Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown.



Step 2: Count the rise; since it goes down, it is negative. The rise is -2.

Step 3: Count the run. The run is 3.

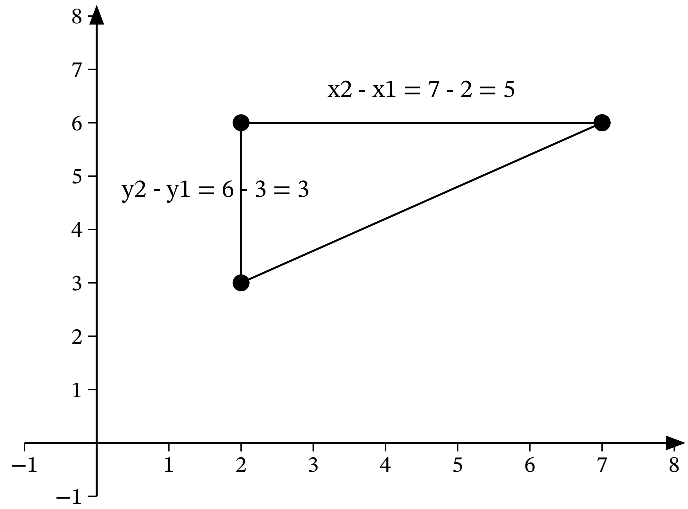
Step 4: Use the slope formula $m = \frac{rise}{run}$ substitute the values of the rise and run. $m = \frac{-2}{3}$
The slope of the line is $-\frac{2}{3}$.

The solution is y decreases by 2 units as x increases by 3 units.

Sometimes we will need to find the slope of a line between two points when we don't have a graph to measure the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but there is a way to find the slope without graphing. First, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish such points. For example, (x_1, y_1) would be said aloud as "x sub 1, y sub 1" and (x_2, y_2) read "x sub 2, y sub 2." The "sub" is a short way of saying "subscript." We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point in our slope equation. If we had more than two points, (if we were finding more than one slope), we could use (x_3, y_3), (x_4, y_4), and so on.

Let's review how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown.



On the graph, we count the rise of 3 and the run of 5. Notice on the graph that that (x_1, y_1) is the point $(2, 3)$ and (x_2, y_2) is the point $(7, 6)$. The rise can be found by subtracting the y-coordinates, 6 and 3, and the run can be found by subtracting the x-coordinates 7 and 2.

$$m = \frac{rise}{run} = \frac{6-3}{7-2} = \frac{3}{5}$$

We have shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \frac{rise}{run}$. We can use this formula to find the slope of a line.

FORMULA

To find the slope of the line between two points (x_1, y_1) and (x_2, y_2), use the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 4 **Finding the Slope of the Line Using Points**

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

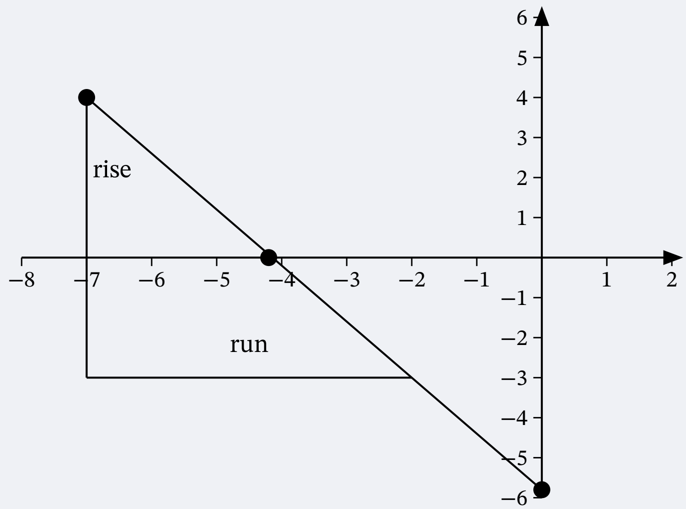
We'll call $(-2, -3)$ point 1 and $(-7, 4)$ point 2.

Step 1: Use the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Step 2: Substitute the values: $m = \frac{4-(-3)}{-7-(-2)}$

Step 3: Simplify: $m = \frac{7}{-5} = -\frac{7}{5}$

Step 4: Verify the slope on the graph shown.



Graphing Functions Using Slope and y-Intercept

We have graphed linear equations by plotting points and using intercepts. Once we see how an equation in slope-intercept form and its graph are related, we will have one more method we can use to graph lines. Review the graph of the equation $y = \frac{1}{2}x + 3$ in and find its slope and y-intercept.

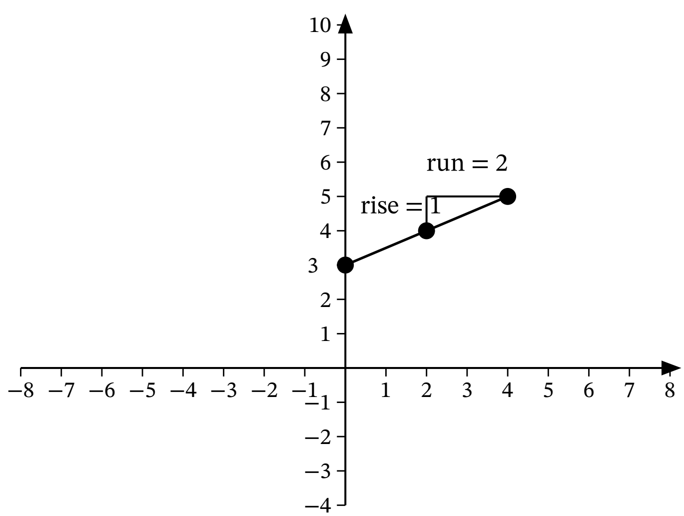


Figure 68: Graph of the equation $y = \frac{1}{2}x + 3$.

The vertical and horizontal lines in the graph show us the rise is 1 and the run is 2, respectively. Substituting into the slope formula: $m = \frac{rise}{run} = \frac{1}{2}$

The y-intercept is $(0, 3)$. Look at the equation of this line.

$$y = \frac{1}{2}x + 3$$

Look at the slope and y-intercept.

$$\text{slope } m = \frac{1}{2} \text{ and } y \text{ - intercept } (0, 3)$$

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the y-coordinate of the y-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in **slope-intercept form**. Sometimes the slope-intercept form is called the y-form.

Example 5 **Finding the Slope and y-Intercept of a Line**

Identify the slope and y-intercept of the line from the equation:

1. $y = -\frac{4}{7}x - 2$
2. $x + 3y = 9$

Solution

1. We compare our equation to the slope-intercept form of the equation.
   Step 1: Write the slope-intercept form of the equation of the line.
   $$y = mx + b$$
   Step 2: Write the equation of the line.
   $$y = -\frac{4}{7}x - 2$$
   Step 3: Identify the slope.
   $$m = -\frac{4}{7}$$
   Step 4: Identify the y-intercept.
   $$y \text{ - intercept is } (0, -2)$$
2. When an equation of a line is not given in slope-intercept form, our first step will be to solve the equation for y.
   Step 1: Solve for y.
   $$x + 3y = 9$$
   Step 2: Subtract x from each side.
   $$3y = -x + 9$$

Step 3: Divide both sides by 3.

$$\frac{3y}{3} = \frac{-x+9}{3}$$

Step 4: Simplify.

$$y = -\frac{1}{3}x + 3$$

Step 5: Write the slope-intercept form of the equation of the line.

$$y = mx + b$$

Step 6: Write the equation of the line.

$$y = -\frac{1}{3}x + 3$$

Step 7: Identify the slope.

$$m = -\frac{1}{3}$$

Step 8: Identify the y-intercept.

$$y\text{ - intercept is } (0, 3)$$

Example 6 **Graphing the Slope and y-Intercept**

Graph the line of the equation $y = -x + 4$ using its slope and y-intercept.

Solution

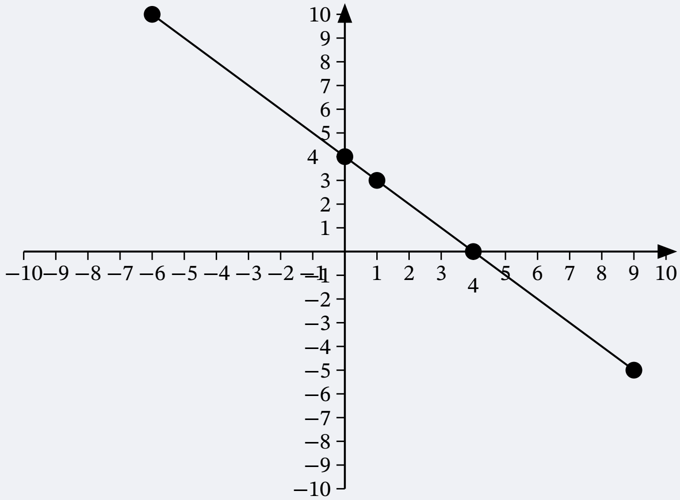
The equation is in slope-intercept form $y = mx + b$.

$$y = -x + 4\,.$$

Step 1: Identify the slope and y-intercept.
$m = -1$, y-intercept is $(0, 4)$.

Step 2: Plot the y-intercept on the coordinate system .

1. Identify the rise over the run.
   $m = -1 = \frac{-1}{1}$
2. Count out the rise and run to mark the second point.
   rise -1, run 1



Graphing Horizontal and Vertical Lines

Some linear equations have only one variable. They may have just x without the y, or just y without an x. This changes how we make a table of values to get the points to plot. Let us consider the equation $x = -3$. This equation has only one variable, x. The equation says that x is always equal to -3, so its value does not depend on y. No matter what the value of y is, the value of x is always -3. To make a table of values, write -3 in for all the x-values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we will use 1, 2, and 3 for the y-coordinates in the table below.

$x = -3$		
x	y	(x, y)
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Plot the points from the table and connect them with a straight line . Notice that we have graphed a vertical line.

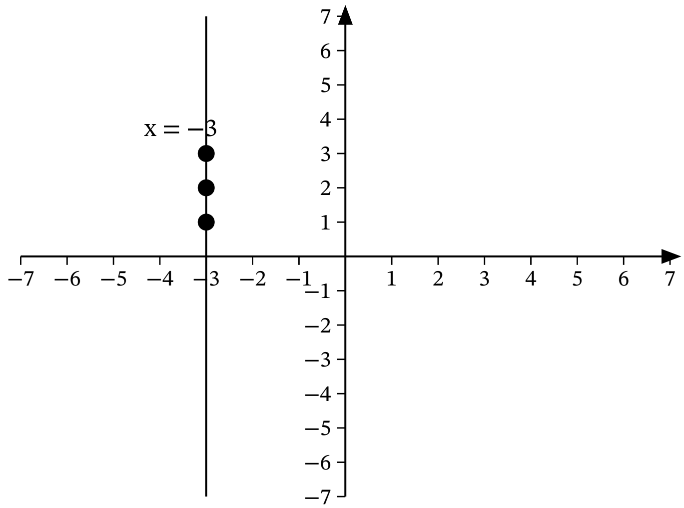


Figure 70: Graph of $x = -3$

What is the slope? If we take the two points $(-3, 3)$ and $(-3, 1)$ then the rise is 2 and the run is 0. Using the slope formula we get: $m = \frac{rise}{run} = \frac{2}{0}$

The slope is **undefined** since division by zero is undefined. We say that the slope of the vertical line $x = -3$ is undefined. The slope of any **vertical line** $x = a$ (where a is any number) will be undefined.

What if the equation has y but no x? Let's graph the equation $y = 4$. This time the y-value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y values in the table below and then choose any values for x. We will use 0, 2, and 4 for the x-coordinates.

$y = 4$		
x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

In , we have graphed a horizontal line passing through the y-axis at 4.

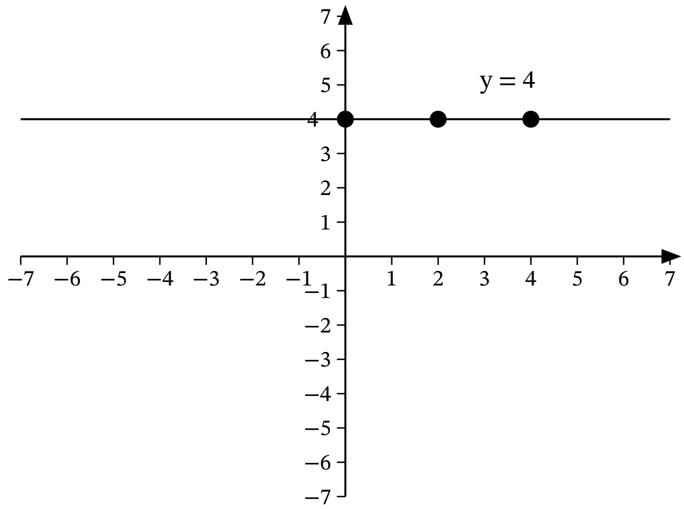


Figure 71: Graph of $y = 4$

What is the slope? If we take the two points $(2, 4)$ and $(4, 4)$ then the rise is 0 and the run is 2. Using the slope formula, we get $m = \frac{rise}{run} = \frac{0}{2} = 0$. The slope of the horizontal line $y = 4$ is 0. The slope of any **horizontal line** $y = b$ (where b is any number) will be 0. When the y-coordinates are the same, the rise is 0.

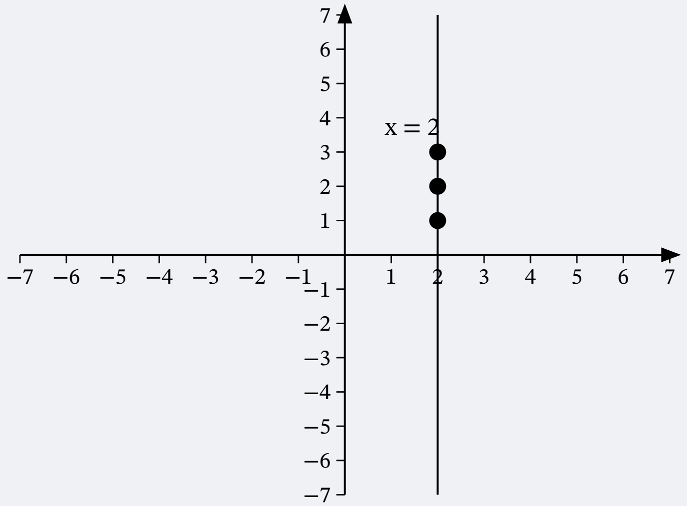
Example 7 **Graphing A Vertical Line**

Graph: $x = 2$.

Solution

The equation has only one variable, x, and x is always equal to 2. We create a table where x is always 2 and then put in any values for y. The graph is a vertical line passing through the x-axis at 2 .

$x = 2$		
x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$



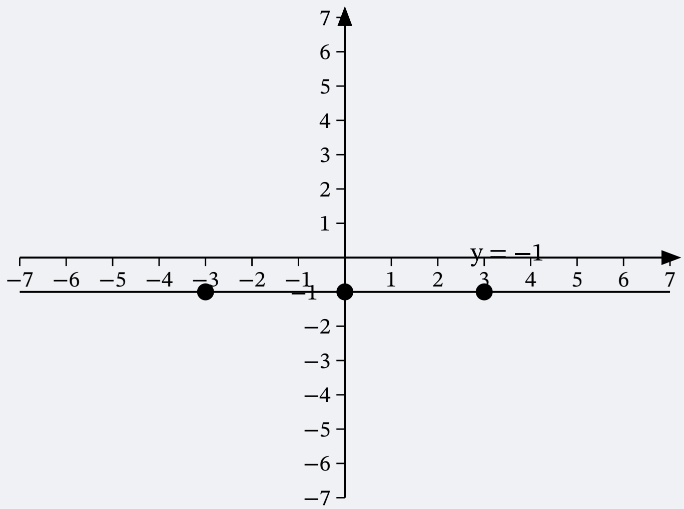
Example 8 **Graphing A Horizontal Line**

Graph: $y = -1$.

Solution

The equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1 .

$y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$



The table below summarizes all the methods we have used to graph lines.

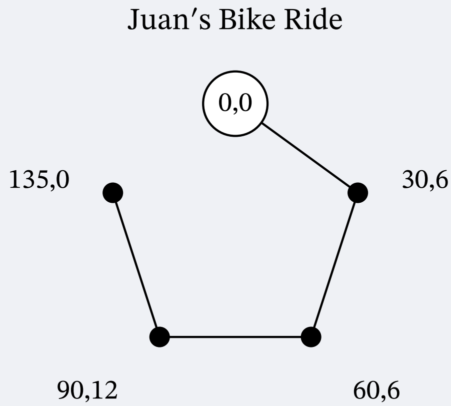
Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y	$y = mx + b$	x | y 0 | | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Interpreting Graphs of Functions

An important yet often overlooked area in algebra involves interpreting graphs. Oftentimes in math classes, students are given mathematical functions and can make graphs to represent them. But the interpretation of graphs is a more applicable skill to the real world. Being able to “read” a graph—understanding its domain and range, what the intercepts mean, and what the slope (or curve) means— that’s a real-world skill.

Example 9 **Interpreting a Graph**

In the x-axis on the graph represents the 120-minute bike ride Juan went on. The y-axis represents how far away he was from his home.



1. Interpret the x- and y-intercept.
2. For each segment, find the slope.
3. Create an interpretation of this graph (i.e., make up a story that goes with it).

Solution

1. $(0, 0)$ is the x- and y-intercept and represents Juan at home before his bike ride. The distance from home is 0 miles and 0 minutes have passed.
2. In the first 30 minutes, the slope is $\frac{1}{5}$ and indicates Juan is traveling 1 mile for every 5 minutes. Between 30 and 60 minutes, the slope is 0 and indicates that he's not riding the bike (the distance is not increasing). Then between 60 and 90 minutes, the slope is $\frac{1}{5}$ again. Finally, after 90 minutes the slope is $-\frac{4}{15}$, meaning Juan is getting 4 miles closer to home every 15 minutes.
3. Answers will vary. Juan left his house for a bike ride. After 30 minutes, he was 6 miles from home and he stopped for ice cream at his local ice-cream truck. He enjoyed his ice cream for 30 minutes. He then jumped back on his bike and rode to his friend's house. He arrived there 30 minutes later. His friend's house was 12 miles from his home. His friend was not home so he immediately turned around and quickly rode home in 45 minutes.

Modeling Applications Using Slope and y-Intercept

Many real-world applications are modeled by linear equations. We will review a few applications here so you can understand how equations written in slope-intercept form relate to real-world situations. Usually when a linear equation model uses real-world data, different letters are used for the variables instead of using only x and y. The variable names often remind us of what quantities are being measured. Also, we often need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

Example 10 **Converting Temperature**

The equation $F = \frac{9}{5}C + 32$ is used to convert temperatures from degrees Celsius (C) to degrees Fahrenheit (F).

1. Find the Fahrenheit temperature for a Celsius temperature of 0°.
2. Find the Fahrenheit temperature for a Celsius temperature of 20°.
3. Interpret the slope and F-intercept of the equation.
4. Graph the equation.

Solution

1. Find the Fahrenheit temperature for a Celsius temperature of 0°.

Find F when $C = 0$.	$F = \frac{9}{5}(0) + 32$
Simplify.	$F = 32$

2. Find the Fahrenheit temperature for a Celsius temperature of 20°.

Find F when $C = 20$.	$F = \frac{9}{5}(20) + 32$
Simplify.	$F = 36 + 32$
Simplify.	$F = 68$

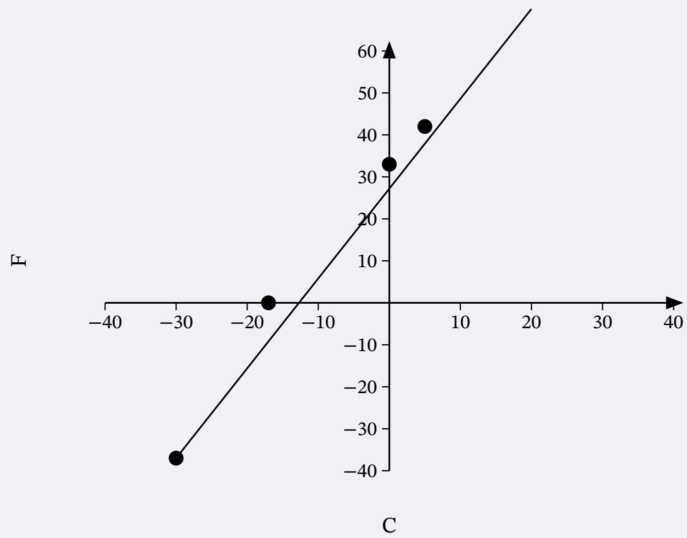
3. Interpret the slope and F-intercept of the equation. Even though this equation uses F and C, it is still in slope-intercept form.

$$y = mx + b$$
$$F = mC + b$$
$$F = \frac{9}{5}C + 32$$

The slope $\frac{9}{5}$ means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees.

The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

1. Graph the equation. We will need to use a larger scale than our usual. Start at the F-intercept $(0, 32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown.



Example 11 **Calculating Driving Costs**

Sam drives a delivery van. The equation $C = 0.5d + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, d, that he drives.

1. Find Sam's cost for a week when he drives 0 miles.
2. Find the cost for a week when he drives 250 miles.
3. Interpret the slope and C-intercept of the equation.
4. Graph the equation.

Solution

1. Find Sam's cost for a week when he drives 0 miles.

Find C when $d = 0$.	$C = 0.5(0) + 60$
Simplify.	$C = 60$

Sam's costs are \$60 when he drives 0 miles.

1. Find the cost for a week when he drives 250 miles.

Find C when $d = 250$.	$C = 0.5(250) + 60$
Simplify.	$C = 185$

Sam's costs are \$185 when he drives 250 miles.

1. Interpret the slope and C-intercept of the equation.

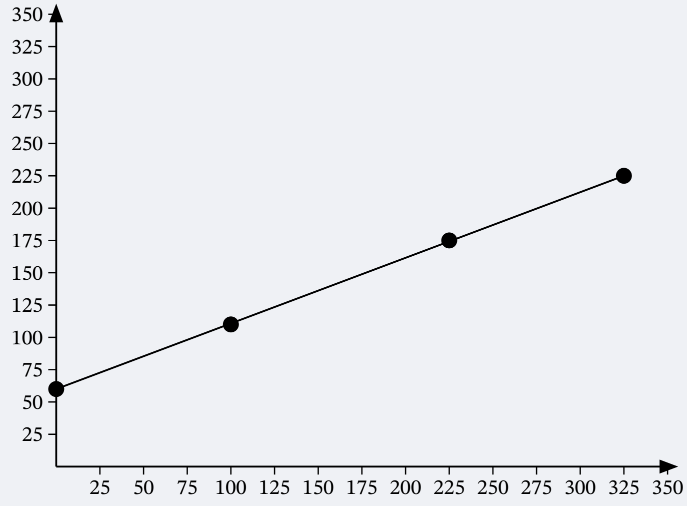
$$y = mx + b$$
$$C = 0.5d + 60$$

The slope, 0.5, means that the weekly cost, C, increases by \$0.50 when the number of miles driven, d, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is \$60.

1. Graph the equation . We'll need to use a larger scale than usual. Start at the C-intercept (0, 60). To count out the slope $m = 0.5$, we rewrite it as an equivalent fraction that will make our graphing easier.

$$m = 0.5 = \frac{5}{10} = \frac{50}{100}$$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be (100, 110).



Key Terms

- intercepts of a line
- slope
- slope-intercept form

Key Concepts

- Every linear function can be graphically represented by a unique line that shows all the solutions of the equation.
- The points where the graph of a line intersects the x-axis and y-axis are called the intercepts of the line.
- Most lines will have one x-intercept and one y-intercept. Only if the line is straight vertical (no y-intercept) or straight horizontal (no x-intercept) will it not have both intercepts. Note

that a line that is straight vertical is not a function, but a line that is straight horizontal is a function.

- Since any two points determine a straight line, any linear function can be graphed if both intercepts are known.
- The slope of a linear function is the ratio of the vertical change divided by the horizontal change. It is often referred to as $\frac{rise}{run}$.
- A formula for finding the slope of linear functions is $\frac{y_2-y_1}{x_2-x_1}$, for any two points of the linear function (x_1, y_1) and (x_2, y_2).

Formulas

- To calculate slope (m), use the formula

$m = \frac{rise}{run}$,

where the rise measures the vertical change and the run measures the horizontal change.

- To find the slope of the line between two points (x_1, y_1) and (x_2, y_2), use the formula $m = \frac{y2-y1}{x2-x1}$

5.9 Systems of Linear Equations in Two Variables

Figure 78: Fruits and vegetables at a farmer's market.

Learning Objectives

After completing this section, you should be able to:

1. Determine and show whether an ordered pair is a solution to a system of equations.
2. Solve systems of linear equations using graphical methods.
3. Solve systems of linear equations using substitution.
4. Solve systems of linear equations using elimination.

5. Identify systems with no solution or infinitely many solutions.
6. Solve applications of systems of linear equations.

In this section, we will learn how to solve systems of linear equations in two variables. There are several real-world scenarios that can be represented by systems of linear equalities. Suppose two friends, Andrea and Bart, go shopping at a farmers market to buy some vegetables. Andrea buys 2 tomatoes and 4 cucumbers and spends \$2.00. Bart buys 4 tomatoes and 5 cucumbers and spends \$2.95. What is the price of each vegetable?

Determining If an Ordered Pair Is a Solution to a System of Equations

When we solved linear equations in Linear Equations in One Variable with Applications and Linear Inequalities in One Variable with Applications, we learned how to solve linear equations with one variable. Now we will work with two or more linear equations grouped together, which is known as a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations in two unknowns (variables) and applications of systems of linear equations. An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, such as $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line. To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the **solutions of a system of equations**.

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 1 **Determining Whether an Ordered Pair Is a Solution to the System**

Determine whether the ordered pair is a solution to the system.

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

1. $(-2, -1)$
2. $(-4, -3)$

Solution

1.

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute $x = -2$ and $y = -1$ into both equations.

$$x - y = -1$$
$$-2 - (-1) \stackrel{?}{=} -1$$
$$-1 = -1 \checkmark \quad 2x - y = -5$$
$$2(-2) - (-1) \stackrel{?}{=} -5$$
$$-3 \neq -5$$

$(-2, -1)$ does not make both equations true. $(-2, -1)$ is not a solution.

2. We substitute $x = -4$ and $y = -3$ into both equations.

$$x - y = -1$$
$$-4 - (-3) \stackrel{?}{=} -1$$
$$-1 = -1 \checkmark \quad 2x - y = -5$$
$$2 \cdot (-4) - (-3) \stackrel{?}{=} -5$$
$$-5 = -5 \checkmark$$

$(-4, -3)$ makes both equations true. $(-4, -3)$ is a solution.

Example 2 **Determining Whether an Ordered Pair Is a Solution to the System**

Determine whether the ordered pair is a solution to the system

$$\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$$

1. $(-4, -5)$
2. $(-4, 5)$

Solution

1. $$\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$$

Substitute -4 for x and -5 for y into both equations.

$$-5 \stackrel{?}{=} \frac{3}{2}(-4) + 1$$
$$-5 \stackrel{?}{=} 3(-2) + 1$$
$$-5 \stackrel{?}{=} -6 + 1$$
$$-5 = -5 \checkmark \quad 2(-4) - 3(-5) \stackrel{?}{=} 7$$
$$(-8) - (-15) \stackrel{?}{=} 7$$
$$-8 + 15 \stackrel{?}{=} 7$$
$$7 = 7 \checkmark$$

$(-4, -5)$ is a solution.

2. $$\begin{cases} y = \frac{3}{2}x + 1 \\ 2x - 3y = 7 \end{cases}$$

Substitute -4 for x and 5 for y into both equations.

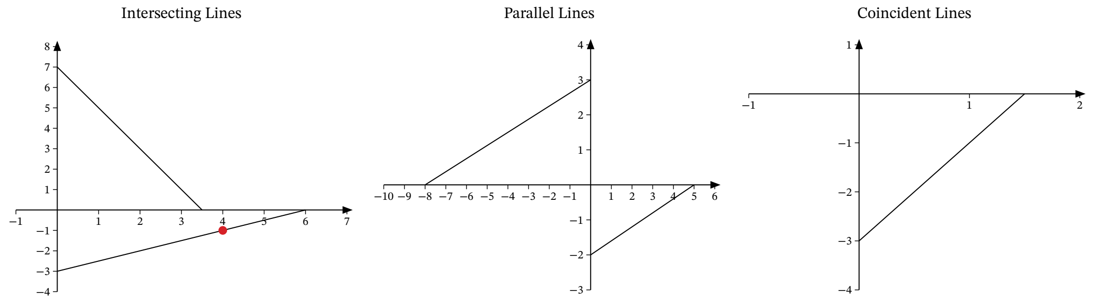
$$5 \stackrel{?}{=} \frac{3}{2}(-4) + 1$$
$$5 \stackrel{?}{=} 3(-2) + 1$$
$$5 \stackrel{?}{=} -6 + 1$$
$$5 \neq -5 \quad 2(-4) - 3(5) \stackrel{?}{=} 7$$
$$(-8) - (15) \stackrel{?}{=} 7$$
$$-8 - 15 \stackrel{?}{=} 7$$
$$-23 \neq 7$$

$(-4, 5)$ is not a solution.

Solving Systems of Linear Equations Using Graphical Methods

We will use three methods to solve a system of linear equations. The first method we will use is graphing. The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what points the lines have in common, we will find the solution to the system.

Most linear equations in one variable have one solution; but for some equations called **contradictions**, there are no solutions, and for other equations called **identities**, all numbers are solutions. Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end you will decide which method was the most convenient way to solve this system.

The steps to use to solve a system of linear equations by graphing are shown here.

Step 1: Graph the first equation.

Step 2: Graph the second equation on the same rectangular coordinate system.

Step 3: Determine whether the lines intersect, are parallel, or are the same line.

Step 4: Identify the solution to the system.

If the lines intersect, identify the point of intersection. This is the solution to the system.

If the lines are parallel, the system has no solution.

If the lines are the same, the system has an infinite number of solutions.

Step 5: Check the solution in both equations.

Example 3 **Solving a System of Linear Equations by Graphing**

Solve this system of linear equations by graphing.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

Solution

Step 1: Graph the first equation.	To graph the first line, write the equation in slope-intercept form. $\begin{aligned} 2x + y &= 7 \\ y &= -2x + 7 \end{aligned}$ $m = -2$ $b = 7$	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2: Graph the second equation on the same rectangular coordinate system.	To graph the second line, use intercepts. $x - 2y = 6$ $(0, -3)$ $(6, 0)$	
Step 3: Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Step 4 and Step 5: Identify the solution to the system. If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. If the lines are parallel, the system has no solution. If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4, -1)$. $\begin{aligned} 2x + y &= 7 \\ 2(4) + (-1) &\stackrel{?}{=} 7 \\ 8 - 1 &\stackrel{?}{=} 7 \\ 7 &= 7 \checkmark \end{aligned}$ $\begin{aligned} x - 2y &= 6 \\ 4 - 2(-1) &\stackrel{?}{=} 6 \\ 6 &= 6 \checkmark \end{aligned}$ The solution is $(4, -1)$.

Solving Systems of Linear Equations Using Substitution

We will now solve systems of linear equations by the substitution method. We will use the same system we used for graphing.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy. Then, we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true. This process is summarized here:

Step 1: Solve one of the equations for either variable.

Step 2: Substitute the expression from Step 1 into the other equation.

Step 3: Solve the resulting equation.

Step 4: Substitute the solution in Step 3 into either of the original equations to find the other variable.

Step 5: Write the solution as an ordered pair.

Step 6: Check that the ordered pair is a solution to both original equations.

Example 4 **Solving a System of Linear Equations Using Substitution**

Solve this system of linear equations by substitution:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

Solution

Step 1: Solve one of the equations for either variable.	We'll solve the first equation for y.	$\begin{cases}2x+y=7\\x-2y=6\end{cases}$ $2x+y=7$ $y=7-2\boldsymbol{x}$
Step 2: Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7-2x$.	$x-2y=6$ $x-2(7-2\boldsymbol{x})=6$
Step 3: Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x-2(7-2x)=6$ $x-14+4x=6$ $5x=20$ $x=4$
Step 4: Substitute the solution from Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x+y=7$ $2(4)+y=7$ $8+y=7$ $y=-1$
Step 5: Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4,-1)$
Step 6: Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into both equations and make sure they are both true.	$2x+y=7$ $2(4)+(-1)\stackrel{?}{=}7$ $7=7\checkmark$ $x-2y=6$ $4-2(-1)\stackrel{?}{=}6$ $6=6\checkmark$ Both equations are true. $(4,-1)$ is the solution to the system.

Solving Systems of Linear Equations Using Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small, and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the elimination method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

The elimination method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal. For any expressions a, b, c, and d:

if $a = b$

and $c = d$

then $a + c = b + d$.

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable. Notice how that works when we add these two equations together:

$$\begin{cases} 3x + y = 5 \\ \underline{2x - y = 0} \end{cases}$$
$$5x \quad = 5$$

The y's add to zero and we have one equation with one variable. Let us try another one:

$$\begin{cases} x + 4y = -2 \\ 2x + 5y = -2 \end{cases}$$

This time we do not see a variable that can be immediately eliminated if we add the equations. But if we multiply the first equation by -2, we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{cases} -2(x + 4y) = -2(-2) \\ 2x + 5y = \quad -2 \end{cases}$$

Then rewrite the system of equations.

$$\begin{cases} -2x - 8y = \quad 4 \\ 2x + 5y = -2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

$$\begin{cases} -2x - 8y = 4 \\ \underline{2x + 5y = -2} \end{cases}$$
$$-3y = 2$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations. Here's a summary of using the elimination method:

Step 1: Write both equations in standard form. If any coefficients are fractions, clear them.

Step 2: Make the coefficients of one variable opposites.

Decide which variable you will eliminate.

Multiply one or both equations so that the coefficients of that variable are opposites.

Step 3: Add the equations resulting from Step 2 to eliminate one variable.

Step 4: Solve for the remaining variable.

Step 5: Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.

Step 6: Write the solution as an ordered pair.

Step 7: Check that the ordered pair is a solution to both original equations.

Example 5 **Solving a System of Linear Equations Using Elimination**

Solve this system of linear equations by elimination:

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

Solution

Step 1: Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Step 2: Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2x + \boldsymbol{y} = 7 \\ x - 2\boldsymbol{y} = 6 \end{cases}$ $\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Step 3: Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x + 2y = 14 \\ \underline{x - 2y = 6} \end{cases}$ $5x \quad = 20$
Step 4: Solve for the remaining variable.	Solve for x.	$x = 4$
Step 5: Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Step 6: Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Step 7: Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true? Yes!	$2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $7 = 7\checkmark$ $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 = 6\checkmark$ The solution is $(4, -1)$.

Identifying Systems with No Solution or Infinitely Many Solutions

In all the systems of linear equations so far, the lines intersected, and the solution was one point. In Example 6 and Example 7, we will look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 6 **Solving a System of Linear Equations with No Solution**

Solve the system by a method of your choice:

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

Solution

Let us solve the system of linear equations by graphing.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

To graph the first equation, we will use its slope and y-intercept.

$$\boldsymbol{y} = \frac{1}{2}\boldsymbol{x} - 3$$

$$m = \frac{1}{2}$$

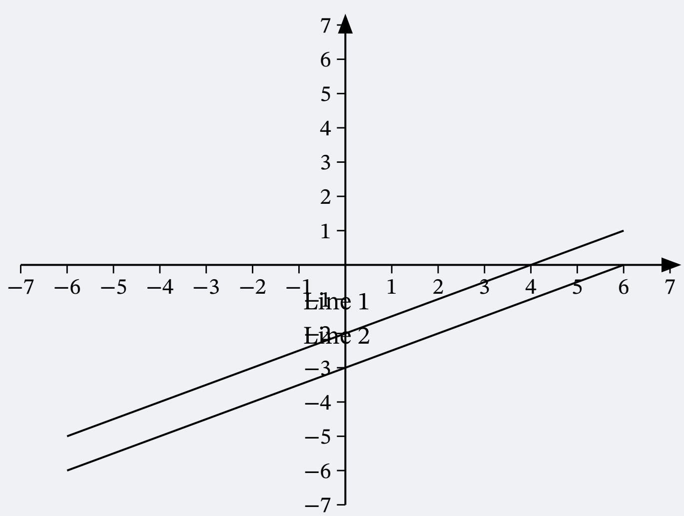
$$b = -3$$

To graph the second equation, we will use the intercepts.

$$\boldsymbol{x} - 2\boldsymbol{y} = 4$$

x	y
0	−2
4	0

Graph the lines .



Determine the points of intersection. The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Example 7 **Solving a System of Linear Equations with Infinite Solutions**

Solve the system by a method of your choice:

$$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$$

Solution

Let us solve the system of linear equations by graphing.

$$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$$

Find the slope and y-intercept of the first equation.

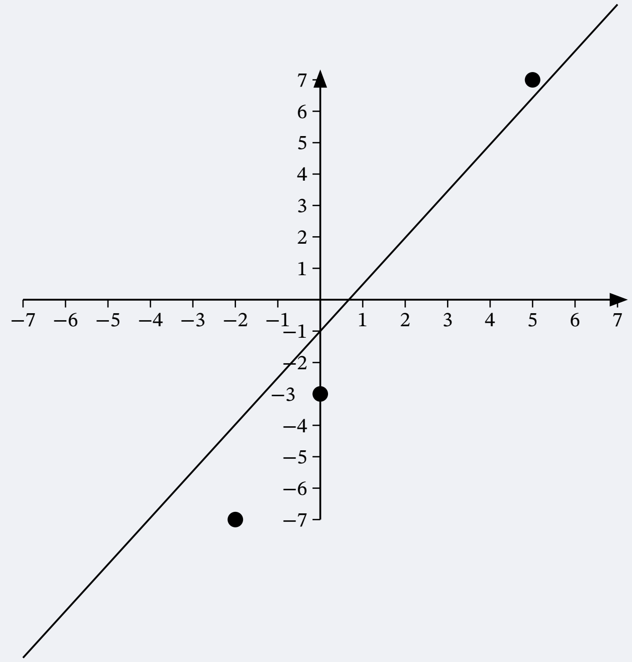
$$\begin{aligned} y &= 2x - 3 \\ m &= 2 \\ b &= -3 \end{aligned}$$

Find the intercepts of the second equation.

$$-6x + 3y = -9$$

x	y
0	-3
$\frac{3}{2}$	0

Graph the lines .

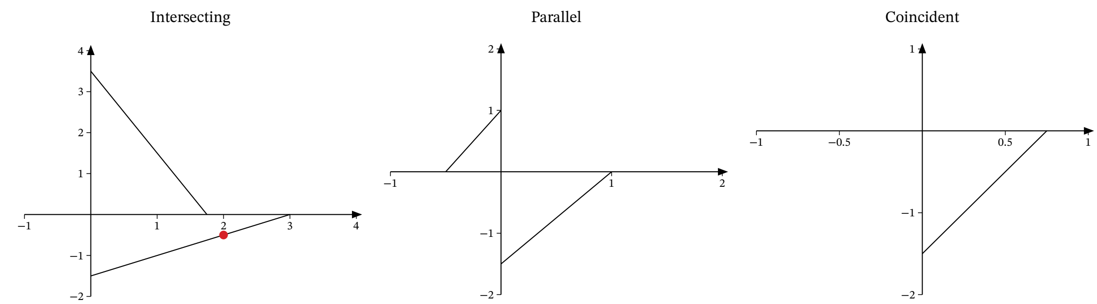


The lines are the same! Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.

There are infinitely many solutions to this system.

In the previous example, if you write the second equation in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept. Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true. There are infinitely many solutions to the system. We say the two lines are coincident. **Coincident lines** have the same slope and same y-intercept. A system of equations that has at least one solution is called a **consistent system**. A system with parallel lines has no solution. We call a system of equations like this an **inconsistent system**. It has no solution.

We also categorize the equations in a system of equations by calling the equations independent or dependent. If two equations are independent, they each have their own set of solutions. Intersecting lines and parallel lines are independent. If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines. Let us sum this up by looking at the graphs of the three types of systems. See and the table that follows



Lines	Intersecting	Parallel	Coincident
Number of Solutions	1 point	No solution	Infinitely many
Consistent/Inconsistent	Consistent	Inconsistent	Consistent
Dependent/Independent	Independent	Independent	Dependent

WORK IT OUT

Using Matrices and Cramer's Rule to Solve Systems of Linear Equations

An m by n matrix is an array with m rows and n columns, where each item in the matrix is a number. Matrices are used for many things, but one thing they can be used for is to represent systems of linear equations. For example, the system of linear equations

$$\{2x + y = 7$$
$$x - 2y = 6$$

can be represented by the following matrix:

$$[21 \quad 7$$
$$1 - 26]$$

To use Cramer's Rule, you need to be able to take the determinant of a matrix. The determinant of a 2 by 2 matrix A, denoted $|A|$, is

$$|A| = |a_{11}a_{12}$$
$$a_{21}a_{22}| = (a_{11} \times a_{22}) + (a_{21} \times a_{12})$$

For example, the determinant of the matrix $|21$ $3-2| = (2 \times -2) - (3 \times 1) = -4 - 3 = -7.$

Cramer's Rule involves taking three determinants:

1. The determinant of the first two columns, denoted $|D|$
2. The determinant of the first column and the third column, denoted $|D_y|$
3. The determinant of the third column and the first column, denoted $|D_x|$.

   Going back to the original matrix $[21 \quad 7$ $1- 26]$

$$|D| = |21$$
$$1-2| = -4-1 = -5$$
$$|D_y| = |27$$
$$16| = 12-7 = 5$$
$$|D_x| = |71$$
$$6-2| = -14-6 = -20$$

Now Cramer's Rule for the solution of the system will be:

$$x = \frac{|D_x|}{|D|}, y = \frac{|D_y|}{|D|}$$

Putting in the values for these determinants, we have $x = \frac{-20}{-5} = 4; y = \frac{5}{-5} = -1$. The solution to the system is the ordered pair $(4, -1)$.

Solving Applications of Systems of Linear Equations

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we will first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

Step 1: Read the problem. Make sure all the words and ideas are understood.

Step 2: Identify what we are looking for.

Step 3: Name what we are looking for. Choose variables to represent those quantities.

Step 4: Translate into a system of equations.

Step 5: Solve the system of equations using good algebra techniques.

Step 6: Check the answer in the problem and make sure it makes sense.

Step 7: Answer the question with a complete sentence.

Example 8 **Applying System to a Real-World Application**

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $25,000 a year plus $15 for each training session. Option B would pay her $10,000 a year plus $40 for each training session. How many training sessions would make the salary options equal?

Solution

Step 1: Read the problem.

Step 2: Identify what we are looking for.

We are looking for the number of training sessions that would make the pay equal.

Step 3: Name what we are looking for.

Let s = Heather's salary, and n = the number of training sessions

Step 4: Translate into a system of equations.

Option A would pay her $25,000 plus $15 for each training session.

$$s = 25,000 + 15n$$

Option B would pay her $10,000 + $40 for each training session.

$$s = 10,000 + 40n$$

The system is shown.

$$\begin{cases} s = 25,000 + 15n \\ s = 10,000 + 40n \end{cases}$$

Step 5: Solve the system of equations.

We will use substitution.

Substitute $25,000 + 15n$ for s in the second equation

$$s = 25,000 + 15n$$
$$s = 10,000 + 40n$$

Solve for n.

$$\begin{aligned} 25,000 + 15\boldsymbol{n} &= 10,000 + 40n \\ 25,000 &= 10,000 + 25n \\ 15,000 &= 25n \\ 600 &= n \end{aligned}$$

Step 6: Check the answer.

Are 600 training sessions a year reasonable?

Are the two options equal when $n = 600$?

Substitute into each equation.

$$s = 25,000 + 15(600) = 34,000$$
$$s = 10,000 + 40(600) = 34,000$$

Step 7: Answer the question.

The salary options would be equal for 600 training sessions.

VIDEO

Practice with Solving Applications of Systems of Equations

Applications of Systems of Linear Equations

Key Terms

- system of linear equations
- solutions of a system of equations
- contradictions

- identities
- coincident lines
- consistent system of linear equations
- inconsistent system of linear equations

Key Concepts

- To solve a system of linear equations means finding the point or points where the two linear equations intersect.
- Two lines can intersect at one point, no points if they are parallel, or every point if they are the same equation.
- Systems of linear equations can be solved by graphing, by using substitution, or by using the elimination method.

Videos

- Practice with Solving Applications of Systems of Equations
- Applications of Systems of Linear Equations

5.10 Systems of Linear Inequalities in Two Variables

Figure 83: Many college students find part-time jobs at places such as coffee shops to help pay for college.

Learning Objectives

After completing this section, you should be able to:

1. Demonstrate whether an ordered pair is a solution to a system of linear inequalities.
2. Solve systems of linear inequalities using graphical methods.
3. Graph systems of linear inequalities.

4. Interpret and solve applications of linear inequalities.

   In this section, we will learn how to solve systems of linear inequalities in two variables. In Systems of Linear Equations in Two Variables, we learned how to solve for systems of linear equations in two variables and found a solution that would work in both equations. We can solve systems of inequalities by graphing each inequality (as discussed in Graphing Linear Equations and Inequalities) and putting these on the same coordinate system. The double-shaded part will be our solution to the system. There are many real-life examples for solving systems of linear inequalities.

   Consider Ming who has two jobs to help her pay for college. She works at a local coffee shop for \$7.50 per hour and at a research lab on campus for \$12 per hour. Due to her busy class schedule, she cannot work more than 15 hours per week. If she needs to make at least \$150 per week, can she work seven hours at the coffee shop and eight hours in the lab?

Determining If an Ordered Pair Is a Solution of a System of Linear Inequalities

The definition of a **system of linear inequalities** is similar to the definition of a system of linear equations. A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true. The solution of a system of linear inequalities is shown as a shaded region in the xy-coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example 1 **Determining Whether an Ordered Pair Is a Solution to a System**

Determine whether the ordered pair is a solution to the system:

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

1. $(-2, 4)$
2. $(3, 1)$

Solution

1. Is the ordered pair $(-2, 4)$ a solution?

   We substitute $\boldsymbol{x} = -2$ and $\boldsymbol{y} = 4$ into both inequalities.

$$x + 4y \geq \quad 10$$
$$-2 + 4(4) \stackrel{?}{\geq} \quad 10$$
$$14 \geq 10 \text{ true } 3x - 2y < 12$$
$$3(-2) - 2(4) \stackrel{?}{<} \quad 12$$
$$-14 < \quad 12 \text{ true}$$

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

1. Is the ordered pair $(3, 1)$ a solution?

We substitute $x = 3$ and $y = 1$ into both inequalities.

$$x + 4y \geq \quad 10$$
$$3 + 4(1) \stackrel{?}{\geq} \quad 10$$
$$7 \geq 10 \text{ false} \quad 3x - 2y < 12$$
$$3(3) - 2(1) \stackrel{?}{<} \quad 12$$
$$7 < \quad 12 \text{ true}$$

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is not a solution to this system.

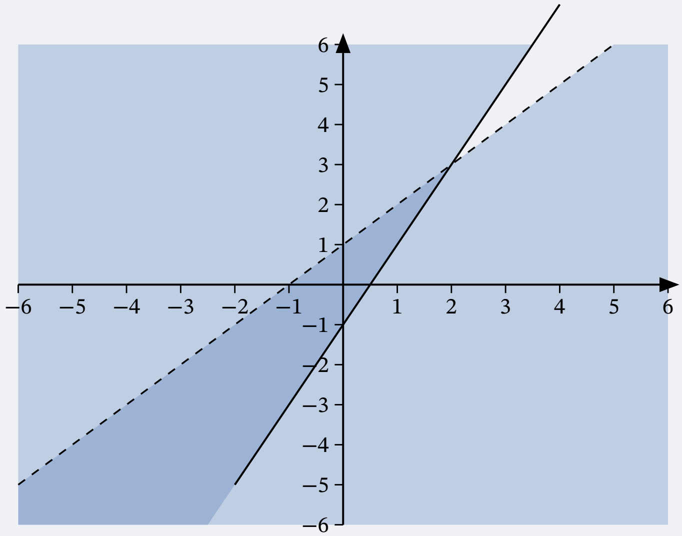
Solving Systems of Linear Inequalities Using Graphical Methods

The solution to a single linear inequality was the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. We will review graphs of linear inequalities and solve the linear inequality from its graph.

Example 2 **Solving a System of Linear Inequalities by Graphing**

Use to solve the system of linear inequalities:

$$\{y \geq 2x - 1$$
$$y < x + 1$$



Solution

To solve the system of linear inequalities we look at the graph and find the region that satisfies BOTH inequalities. To do this we pick a test point and check. Let's us pick $(-1, -1)$.

Is $(-1, -1)$ a solution to $y \geq 2x - 1$?

$$\begin{aligned} -1 &\overset{?}{\geq} 2(-1) - 1 \\ -1 &\geq \quad -3 \text{ true} \end{aligned}$$

Is $(-1, -1)$ a solution to $y < x + 1$?

$$\begin{aligned} -1 &\overset{?}{<} -1 + 1 \\ -1 &< \quad 0 \text{ true} \end{aligned}$$

The region containing $(-1, -1)$ is the solution to the system of linear inequalities. Notice that the solution is all the points in the area shaded twice, which appears as the darkest shaded region.

Graphing Systems of Linear Inequalities

We learned that the solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region by graphing, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

Step 1: Graph the first inequality.

Graph the boundary line.

Shade in the side of the boundary line where the inequality is true.

Step 2: On the same grid, graph the second inequality.

Graph the boundary line.

Shade in the side of that boundary line where the inequality is true.

Step 3: The solution is the region where the shading overlaps.

Step 4: Check by choosing a test point.

Example 3 **Solving a System of Linear Inequalities by Graphing**

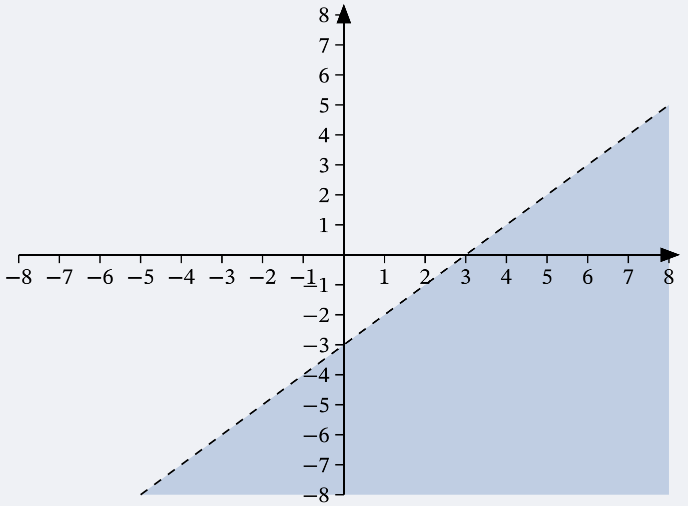
Solve the system by graphing:

$$\{x - y > 3$$

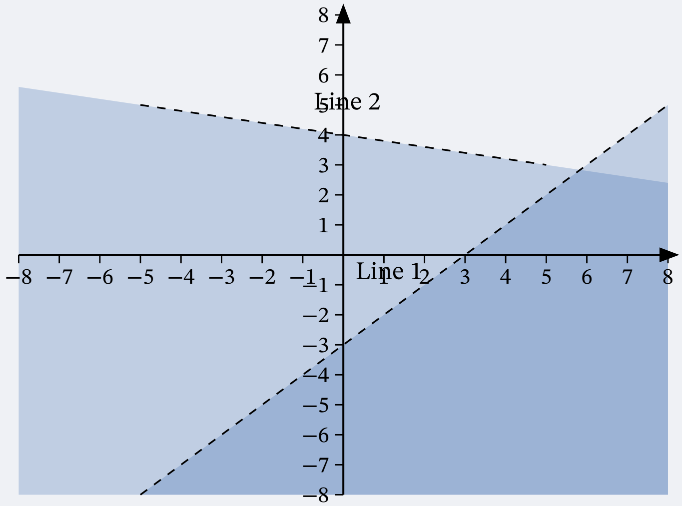
$$y < -\frac{1}{5}x + 4$$

Solution

Graph $x - y > 3$ by graphing $x - y = 3$ and testing a point . The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed. Test $(0, 0)$ which makes the inequality false so shade the side that does not contain $(0, 0)$.



Graph $y < -\frac{1}{5}x + 4$ by graphing $y = -\frac{1}{5}x + 4$ using the slope $m = -\frac{1}{5}$ and y-intercept $b = 4$. The boundary line will be dashed. Test $(0, 0)$ which makes the inequality true, so shade the side that contains $(0, 0)$.



Choose a test point in the solution and verify that it is a solution to both inequalities. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice—which appears as the darkest shaded region.

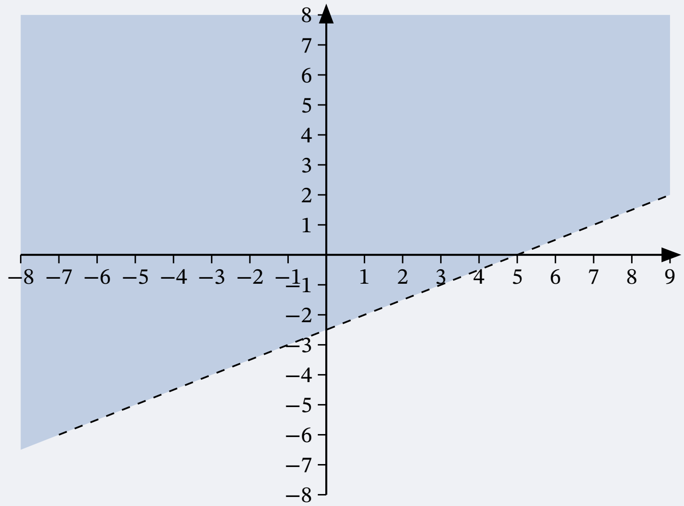
Example 4 **Graphing a System of Linear Inequalities**

Solve the system by graphing:

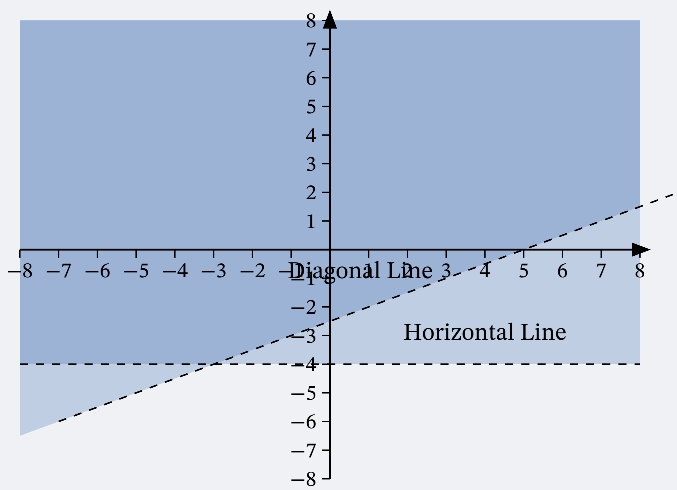
$$\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$$

Solution

Graph $x - 2y < 5$ by graphing $x - 2y = 5$ and testing a point. The intercepts are $x = 5$ and $y = -2.5$ and the boundary line will be dashed. Test $(0, 0)$, which makes the inequality true, so shade the side that contains $(0, 0)$.



Graph $y > -4$ by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$. The boundary line will be dashed. Test $(0, 0)$, which makes the inequality true so shade the side that contains $(0, 0)$.



The point $(0, 0)$ is in the solution, and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice, which appears as the darkest shaded region.

Systems of linear inequalities where the boundary lines are parallel might have no solution. We will see this in the next example.

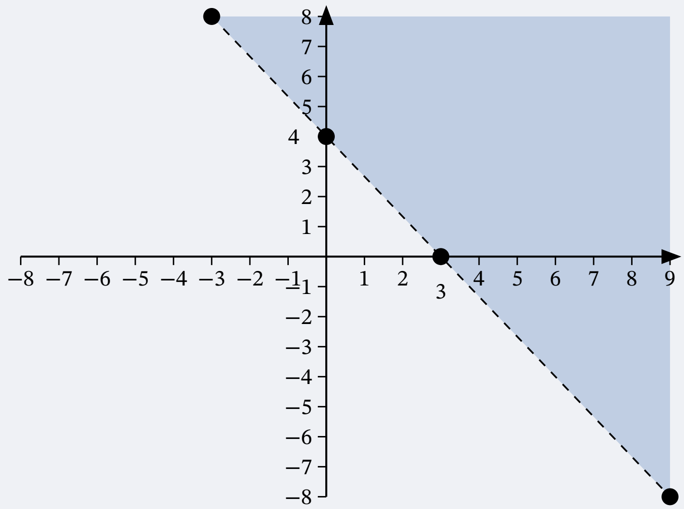
Example 5 **Graphing Parallel Boundary Lines with No Solution**

Solve the system by graphing:

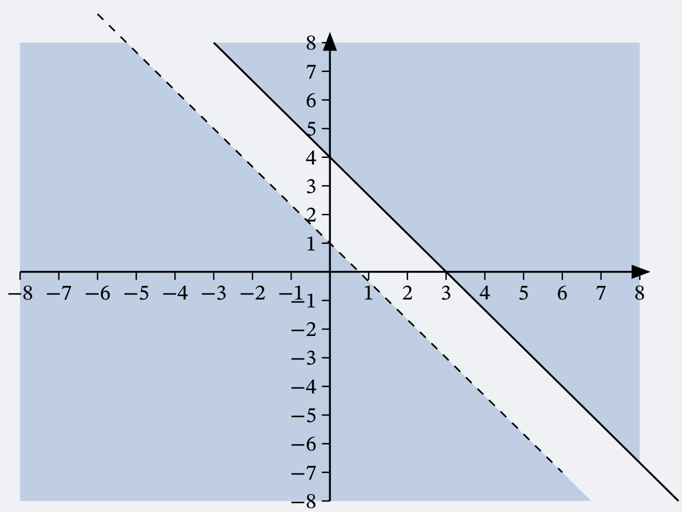
$$\begin{cases} 4x + 3y \geq 12 \\ y < -\frac{4}{3}x + 1 \end{cases}$$

Solution

Graph $4x + 3y \geq 12$, by graphing $4x + 3y = 12$ and testing a point. The intercepts are $x = 3$ and $y = 4$ and the boundary line will be solid. Test $(0, 0)$, which makes the inequality false, so shade the side that does not contain $(0, 0)$.



Graph $y < -\frac{4}{3}x + 1$ by graphing $y = -\frac{4}{3}x + 1$ using the slope $m = -\frac{4}{3}$ and y-intercept $b = 1$. The boundary line will be dashed. Test $(0, 0)$, which makes the inequality true, so shade the side that contains $(0, 0)$.



No shared point exists in both shaded regions, so the system has no solution.

Some systems of linear inequalities where the boundary lines are parallel will have a solution. We will see this in the next example.

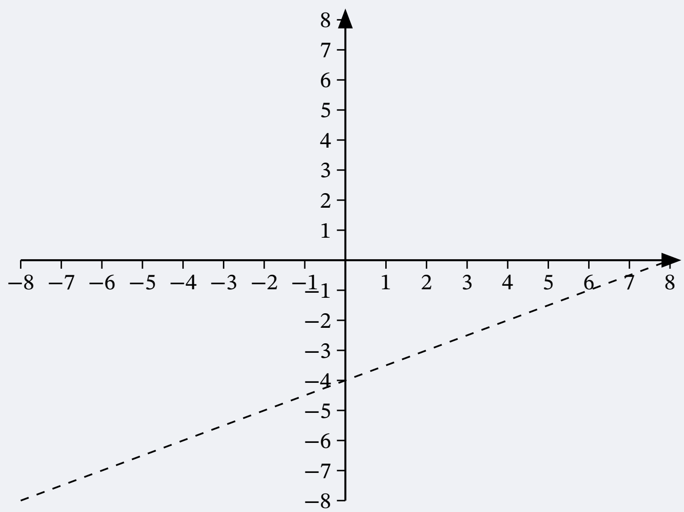
Example 6 **Graphing Parallel Boundary Lines with a Solution**

Solve the system by graphing:

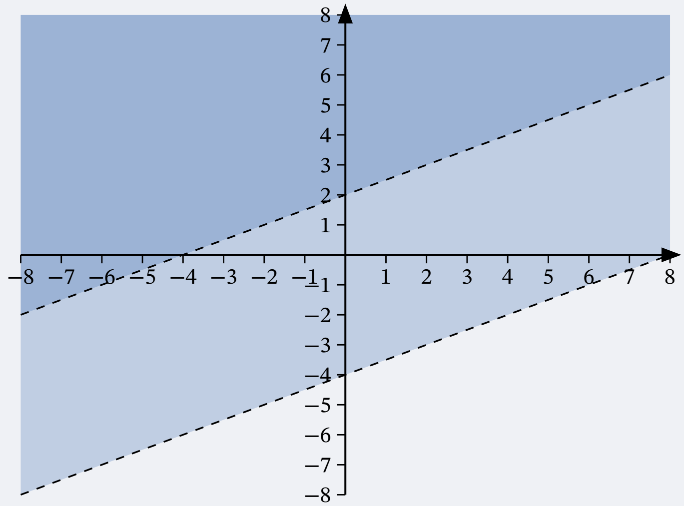
$$\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$$

Solution

Graph $y > \frac{1}{2}x - 4$ by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and the y-intercept $b = -4$. The boundary line will be dashed. Test $(0, 0)$, which makes the inequality true, so shade the side that contains $(0, 0)$.



Graph $x - 2y < -4$ by graphing $x - 2y = -4$ and testing a point. The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed. Choose a test point in the solution and verify that it is a solution to both inequalities. Test $(0, 0)$, which makes the inequality false, so shade the side that does not contain $(0, 0)$.



No point on the boundary lines is included in the solution as both lines are dashed. The solution is the region that is shaded twice which is also the solution to $x - 2y < -4$.

Interpreting and Solving Applications of Linear Inequalities

When solving applications of systems of inequalities, first translate each condition into an inequality. Then graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so add inequalities to the system as additional requirements.

Example 7 **Applying Linear Inequalities to Calculating Photo Costs**

A photographer sells their prints at a booth at a street fair. At the start of the day, they want to have at least 25 photos to display at their booth. Each small photo they display costs \$4 and each large photo costs \$10. They do not want to spend more than \$200 on photos to display.

1. Write a system of inequalities to model this situation.
2. Graph the system.
3. Could they display 10 small and 20 large photos?
4. Could they display 20 large and 10 small photos?

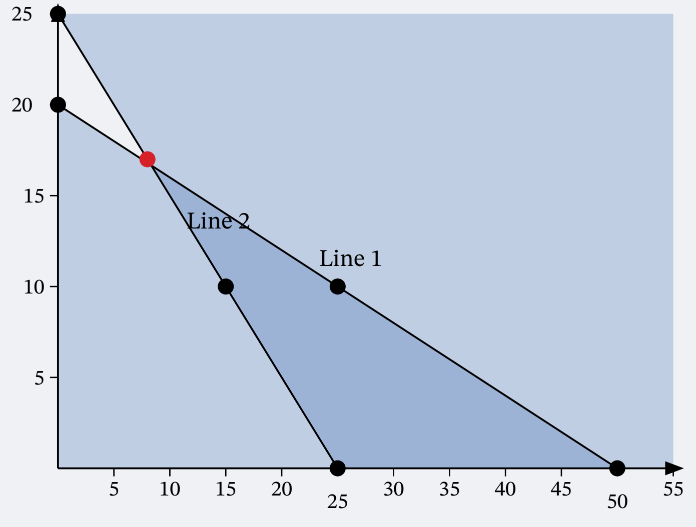
Solution

1. Let $x =$ the number of small photos and $y =$ the number of large photos. To find the system of equations translate the information. They want to have at least 25 photos.

The number of small plus the number of large should be at least 25.	$x + y \geq 25$
\$4 for each small and \$10 for each large must be no more than \$200	$4x + 10y \leq 200$
The number of small photos must be greater than or equal to 0.	$x \geq 0$
The number of large photos must be greater than or equal to 0.	$y \geq 0$
We have our system of equations.	$\begin{cases} x + y \geq 25 \\ 4x + 10y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$

2. Since $x \geq 0$ and $y \geq 0$ (both are greater than or equal to) all solutions will be in the first quadrant. As a result, our graph shows only Quadrant I. To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line. Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade the side that does not include the point $(0, 0)$.

To graph $4x + 10y \leq 200$, graph $4x + 10y = 200$ as a solid line. Choose $(0, 0)$ as a test point. Since it does make the inequality true, shade (bottom left) the side that include the point $(0, 0)$.



The solution of the system is the region of that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(25, 0)$ to $(55, 0)$.

1. To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is

a solution of both equations. It is not, so the photographer would not display 10 small and 20 large photos.

2. To determine if 20 small and 10 large photos would work, we look at the graph to see if the point $(20, 10)$ is in the solution region. We could also test the point to see if it is a solution of both equations. It is, so the photographer could choose to display 20 small and 10 large photos. Notice that we could also test the possible solutions by substituting the values into each inequality.

VIDEO

Solving Systems of Linear Inequalities by Graphing

Systems of Linear Inequalities

Key Terms

- system of linear inequalities

Key Concepts

- To solve a system of linear inequalities means to find the area(s) where the points in that area make all the linear inequalities true.
- Systems of linear inequalities can be solved by graphing the linear equations associated with the inequalities, then 'testing' points to see whether the values of the point make the equation true or not.

Videos

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

5.11 Linear Programming

Figure 94: The aftermath of an earthquake and tsunami.

Learning Objectives

After completing this section, you should be able to:

1. Compose an objective function to be minimized or maximized.
2. Compose inequalities representing a system application.
3. Apply linear programming to solve application problems.

Imagine you hear about some natural disaster striking a far-away country; it could be an earthquake, a fire, a tsunami, a tornado, a hurricane, or any other type of natural disaster. The survivors of this disaster need help—they especially need food, water, and medical supplies. You work for a company that has these supplies, and your company has decided to help by flying the needed supplies into the disaster area. They want to maximize the number of people they can help. However, there are practical constraints that need to be taken into consideration; the size of the airplanes, how much weight each airplane can carry, and so on. How do you solve this dilemma? This is where linear programming comes into play. **Linear programming** is a mathematical technique to solve problems involving finding maximums or minimums where a linear function is limited by various constraints.

As a field, linear programming began in the late 1930s and early 1940s. It was used by many countries during World War II; countries used linear programming to solve problems such as maximizing troop effectiveness, minimizing their own casualties, and maximizing the damage they could inflict upon the enemy. Later, businesses began to realize they could use the concept of linear programming to maximize output, minimize expenses, and so on. In short, linear programming is a method to solve problems that involve finding a maximum or minimum where a linear function is constrained by various factors.

WHO KNEW?

A Mathematician Invents a "Tsunami Cannon"

On December 26, 2004, a massive earthquake occurred in the Indian Ocean. This earthquake, which scientists estimate had a magnitude of 9.0 or 9.1 on the Richter Scale, set off a wave of tsunamis across the Indian Ocean. The waves of the tsunami averaged over 30 feet (10 meters) high, and caused massive damage and loss of life across the coastal regions bordering the Indian Ocean.

Usama Kadri works as an applied mathematician at Cardiff University in Wales. His areas of research include fluid dynamics and non-linear phenomena. Lately, he has been focusing his research on the early detection and easing of the effects of tsunamis. One of his theories involves deploying a series of devices along coastlines which would fire acoustic-gravity waves (AGWs) into an oncoming tsunami, which in theory would lessen the force of the tsunami. Of course, this is all in theory, but Kadri believes it will work. There are issues with creating such a device: they would take a tremendous amount of electricity to generate an AGW, for instance, but if it would save lives, it may well be worth it.

Compose an Objective Function to Be Minimized or Maximized

An **objective function** is a linear function in two or more variables that describes the quantity that needs to be maximized or minimized.

Example 1 **Composing an Objective Function for Selling Two Products**

Miriam starts her own business, where she knits and sells scarves and sweaters out of high-quality wool. She can make a profit of \$8 per scarf and \$10 per sweater. Write an objective function that describes her profit.

Solution

Let x represent the number of scarves sold, and let y represent the number of sweaters sold. Let P represent profit. Since each scarf has a profit of \$8 and each sweater has a profit of \$10, the objective function is $P = 8x + 10y$.

Example 2 **Composing an Objective Function for Production**

William's factory produces two products, widgets and wadgets. It takes 24 minutes for his factory to make 1 widget, and 32 minutes for his factory to make 1 wadget. Write an objective function that describes the time it takes to make the products.

Solution

Let x equal the number of widgets made; let y equal the number of wadgets made; let T represent total time. The objective function is $T = 24x + 32y$.

Composing Inequalities Representing a System Application

For our two examples of profit and production, in an ideal world the profit a person makes and/or the number of products a company produces would have no restrictions. After all, who wouldn't want to have an unrestricted profit? However in reality this is not the case; there are usually several variables that can restrict how much profit a person can make or how many products a company can produce. These restrictions are called **constraints**.

Many different variables can be constraints. When making or selling a product, the time available, the cost of manufacturing and the amount of raw materials are all constraints. In the opening scenario with the tsunami, the maximum weight on an airplane and the volume of cargo it can carry would be constraints. Constraints are expressed as linear inequalities; the list of constraints defined by the problem forms a system of linear inequalities that, along with the objective function, represent a system application.

Example 3 **Representing the Constraints for Selling Two Products**

Two friends start their own business, where they knit and sell scarves and sweaters out of high-quality wool. They can make a profit of \$8 per scarf and \$10 per sweater. To make a scarf, 3 bags of knitting wool are needed; to make a sweater, 4 bags of knitting wool are needed. The friends can only make 8 items per day, and can use not more than 27 bags of knitting wool per day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for profit and the two constraints.

Solution

Let x represent the number of scarves sold, and let y represent the number of sweaters sold. There are two constraints: the number of items the business can make in a day (a maximum of 8) and the number of bags of knitting wool they can use per day (a maximum of 27). The first constraint (total number of items in a day) is written as:

$$x + y \leq 8$$

Since each scarf takes 3 bags of knitting wool and each sweater takes 4 bags of knitting wool, the second constraint, total bags of knitting wool per day, is written as:

$$3x + 4y \leq 27$$

In summary, here are the equations that represent the new business:

$P = 8x + 10y$ This is the profit equation: The business makes \$8 per scarf and \$10 per sweater.

$$\begin{aligned} x + y &\leq 8 \\ 3x + 4y &\leq 27 \end{aligned}$$

Example 4 **Representing Constraints for Production**

A factory produces two products, widgets and wadgets. It takes 24 minutes for the factory to make 1 widget, and 32 minutes for the factory to make 1 wadget. Research indicates that long-term demand for products from the factory will result in average sales of 12 widgets per day and 10 wadgets per day. Because of limitations on storage at the factory, no more than 20 widgets or 17 wadgets can be made each day. Write the inequalities that represent the constraints. Then summarize what has been described thus far by writing the objective function for time and the two constraints.

Solution

Let x equal the number of widgets made; let y equal the number of wadgets made. Based on the long-term demand, we know the factory must produce a minimum of 12 widgets and 10 wadgets per day. We also know because of storage limitations, the factory cannot produce more than 20 widgets per day or 17 wadgets per day. Writing those as inequalities, we have:

$$x \geq 12$$
$$y \geq 10$$
$$x \leq 20$$
$$y \leq 17$$

The number of widgets made per day must be between 12 and 20, and the number of wadgets made per day must be between 10 and 17. Therefore, we have:

$$12 \leq x \leq 20$$
$$10 \leq y \leq 17$$

The system is:

$$T = 24x + 32y$$
$$12 \leq x \leq 20$$
$$10 \leq y \leq 17$$

T is the variable for time; it takes 24 minutes to make a widget and 32 minutes to make a wadget.

Applying Linear Programming to Solve Application Problems

There are four steps that need to be completed when solving a problem using linear programming. They are as follows:

Step 1: Compose an objective function to be minimized or maximized.

Step 2: Compose inequalities representing the constraints of the system.

Step 3: Graph the system of inequalities representing the constraints.

Step 4: Find the value of the objective function at each corner point of the graphed region.

The first two steps you have already learned. Let's continue to use the same examples to illustrate Steps 3 and 4.

Example 5 **Solving a Linear Programming Problem for Two Products**

Three friends start their own business, where they knit and sell scarves and sweaters out of high-quality wool. They can make a profit of \$8 per scarf and \$10 per sweater. To make a scarf, 3 bags of knitting wool are needed; to make a sweater, 4 bags of knitting wool are needed. The friends can only make 8 items per day, and can use not more than 27 bags of knitting wool per day. Determine the number of scarves and sweaters they should make each day to maximize their profit.

Solution

Step 1: Compose an objective function to be minimized or maximized. From Example 3, the objective function is $P = 8x + 10y$.

Step 2: Compose inequalities representing the constraints of the system. From Example 3, the constraints are $x + y \leq 8$ and $3x + 4y \leq 27$.

Step 3: Graph the system of inequalities representing the constraints. Using methods discussed in Graphing Linear Equations and Inequalities, the graphs of the constraints are shown below. Because the number of scarves (x) and the number of sweaters (y) both must be non-negative numbers (i.e., $x \geq 0$ and $y \geq 0$), we need to graph the system of inequalities in Quadrant I only. shows each constraint graphed on its own axes, while shows the graph of the system of inequalities (the two constraints graphed together). In , the large shaded region represents the area where the two constraints intersect. If you are unsure how to graph these regions, refer back to Graphing Linear Equations and Inequalities.

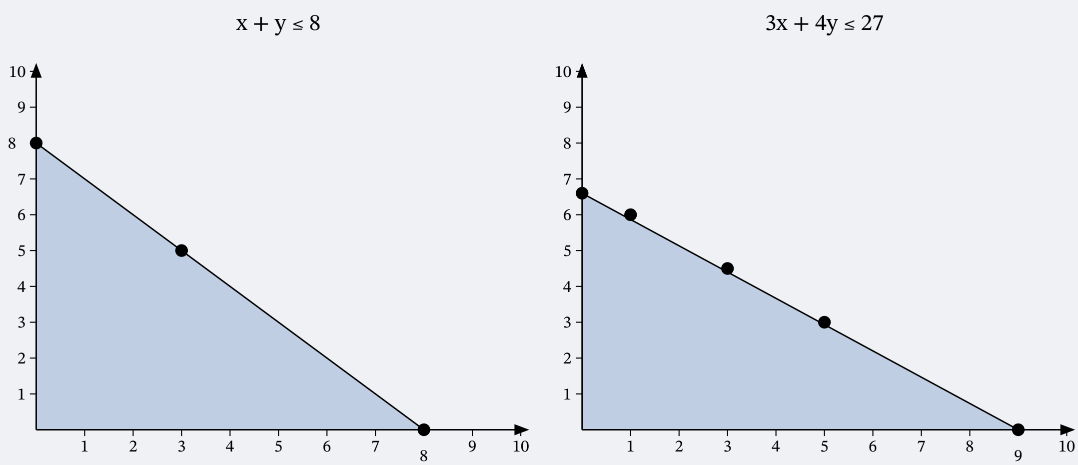


Figure 95: Graphs of each constraint

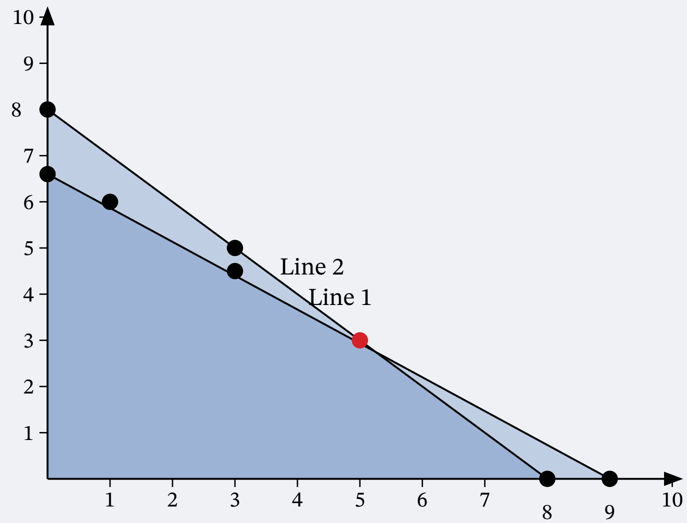


Figure 96: Graph of the System of Inequalities

Step 4: Find the value of the objective function at each corner point of the graphed region. The “graphed region” is the area where both of the regions intersect; in , it is the large shaded area. The “corner points” refer to each vertex of the shaded area. Why the corner points? Because the maximum and minimum of every objective function will occur at one (or more) of the corner points. shows the location and coordinates of each corner point.

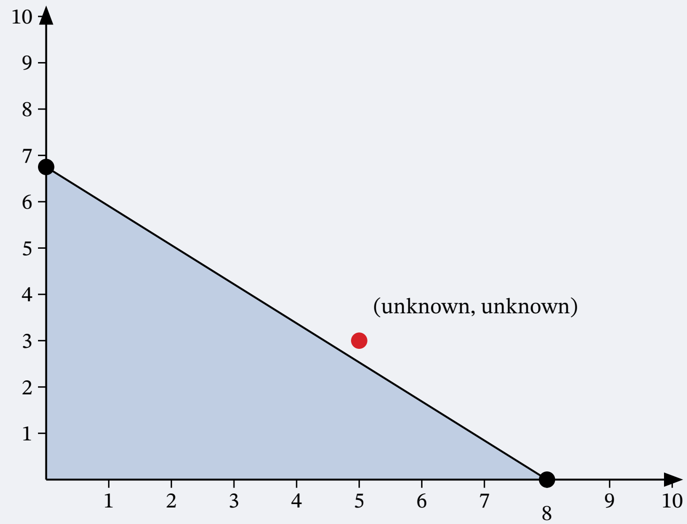


Figure 97: Graph of Region with Corner Points

Three of the four points are readily found, as we used them to graph the regions; the fourth point, the intersection point of the two constraint lines, will have to be found using methods discussed in Systems of Linear Equations in Two Variables, either using substitution or elimination. As a reminder, set up the two equations of the constraint lines:

$$\begin{aligned} 3x + 4y &= 27 \\ x + y &= \ \ 8 \end{aligned}$$

For this example, substitution will be used.

$$\begin{aligned} x + y &= \quad 8 \\ y &= 8 - x\,. \end{aligned}$$

Substituting $8 - x$ into the first equation for y, we have

$$\begin{aligned} 3x + 4(8 - x) &= \ \ 27 \\ 3x + 32 - 4x &= \ \ 27 \\ -x &= -5 \\ x &= \ \ \ 5 \end{aligned}$$

Now, substituting the 5 in for x in either equation to solve for y. Choosing the second equation, we have:

$$\begin{aligned} 5 + y &= 8 \\ y &= 3 \end{aligned}$$

Therefore, $x = 5$, and $y = 3$.

To find the value of the objective function, $P = 8x + 10y$, put the coordinates for each corner point into the equation and solve. The largest solution found when doing this will be the maximum value, and thus will be the answer to the question originally posed: determining the number of scarves and sweaters the new business should make each day to maximize their profit.

Corner (x, y)	Objective Function $P = 8x + 10y$
$(0, 0)$	$P = 8(0) + 10(0) = 0$
$(0, 6.75)$	$P = 8(0) + 10(6.75) = 67.5$
$(5, 3)$	$P = 8(5) + 10(3) = 40 + 30 = 70$
$(8, 0)$	$P = 8(8) + 10(0) = 64$

The maximum value for the profit P occurs when $x = 5$ and $y = 3$. This means that to maximize their profit, the new business should make 5 scarves and 3 sweaters every day.

PEOPLE IN MATHEMATICS

Leonid Kantorovich

Leonid Vitalyevich Kantorovich was born January 19, 1912, in St. Petersburg, Russia. Two major events affected young Leonid's life: when he was five, the Russian Revolution began, making life in St. Petersburg very difficult; so much so that Leonid's family fled to Belarus for a year. When Leonid was 10, his father died, leaving his mother to raise five children on her own.

> Despite the hardships, Leonid showed incredible mathematical ability at a young age. When he was only 14, he enrolled in Leningrad State University to study mathematics. Four years later, at age 18, he graduated with what would be equivalent to a Ph.D. in mathematics.
>
> Although his primary interests were in pure mathematics, in 1938 he began working on problems in economics. Supposedly, he was approached by a local plywood manufacturer with the following question: how to come up with a work schedule for eight lathes to maximize output, given the five different kinds of plywood they had at the factory. By July 1939, Leonid had come up with a solution, not only to the lathe scheduling problem but to other areas as well, such as an optimal crop rotation schedule for farmers, minimizing waste material in manufacturing, and finding optimal routes for transporting goods. The technique he discovered to solve these problems eventually became known as linear programming. He continued to use this technique for solving many other problems involving optimization, which resulted in the book *The Best Use of Economic Resources*, which was published in 1959. His continued work in linear programming would ultimately result in him winning the Nobel Prize of Economics in 1975.

Key Terms

- linear programming
- objective function
- constraint

Key Concepts

- Linear programming is a mathematical technique to solve problems involving finding maximums or minimums where a linear function is limited by various constraints.
- An objective function is a linear function in two or more variables that describes the quantity that needs to be maximized or minimized.
- In linear programming, a constraint is a restriction that affects the maximum or minimum values of an objective function.
- Through the creation of objective functions and restraints, a linear system can be developed and solved through linear programming.

Projects

Ratio and Proportion—Comparing Prices, Part 1

Go to your favorite coffee shops and find out what a same sized drink costs at each. You can do something similar for pizza as well. Find the unit rate (i.e., price per ounce or price per square inch). For example, go to your favorite coffee place and find the price per units on all their large coffee drinks. Or go to your favorite pizza place and compare prices of all their extra-large pizzas (by price per square inch). Write a report on the best deals.

Ratio and Proportion—Comparing Prices, Part 2

Rather than comparing prices of different, but same sized drinks (or pizzas), compare unit prices of the same drinks but of different sizes. Find out what the best bargain is based on price per ounce, price per square inch, etc. For example, compare the prices of your favorite soft drink sold at a local store, but in various sizes (i.e., 12-ounce can, 16-ounce bottle, 20-ounce bottle, 1-liter bottle, and multipacks). Or go to a pizza place and find out what the best bargain is on their menu, based on price per square inch of pizza. Write a report on the best deals.

Systems of Linear Inequalities—Comparing Cell Phone Plans

Go to the websites of different cell phone companies and compare their plans. Write a report on “the best deals. “Best Deals” doesn’t necessarily mean “cheapest.” You will need to look at what each company provides concerning restrictions (constraints) on minutes to talk. What are the constraints on the cell phone coverage for each company? Do they cover your area of the country well? Do they cover the entire United States well, or at least areas where you will be travelling? Is this coverage 5G, or is it less? Can you add a phone easily? Can you bring your previous phone number to this plan? The possibilities of constraints affecting each plan are several. So your task is to determine which plan is best, based on not only cost but also all constraints you deem important.