Chapter 4: Number Representation and Calculation

Figure 1: Different cultures developed different ways to record quantity.

Right now, almost all cultures use the familiar Hindu-Arabic numbering system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 along with place values based on powers of ten. This is a relatively recent development. The system didn't develop until the 6th or 7th century C.E. and took some time to spread across the world, which means other cultures at other times had to develop their own methods of counting and recording quantity. Being different cultures and different times means there were significant differences in counting systems. Cultures needed to count and measure time for agriculture and for religious observations. It was needed for trade. Some languages had words only for one, two, and many. Other cultures developed more complex ways to represent quantity, with the Oksapmin people of New Guinea using an astonishing 27 words for their system.

Representing these quantities in a recorded form likely began with a simple marking system, where one scratch on a stick or bone represented one of whatever was being counted. We still see this today with tally marks. These systems use repeated symbols to represent more than one. We also have systems where different symbols represent different quantities but still use some repetition, such as in Roman numerals.

Other systems were devised that rely on place values, like the Hindu-Arabic system in use today. Place value systems needed a zero, though, and weren't immediately recognized and took time to develop. And within these positional systems there is variation. Some systems counted in twenties, others in tens, and some in a mix (adding another reason to visit Hawaii). Even now, though we all use and think using tens, computers are designed to work in groups of two, which requires a different perspective on numbering.

In this chapter, we explore different numbering systems and grouping systems, eventually discussing base 2, the language of computers.

4.1 Hindu-Arabic Positional System

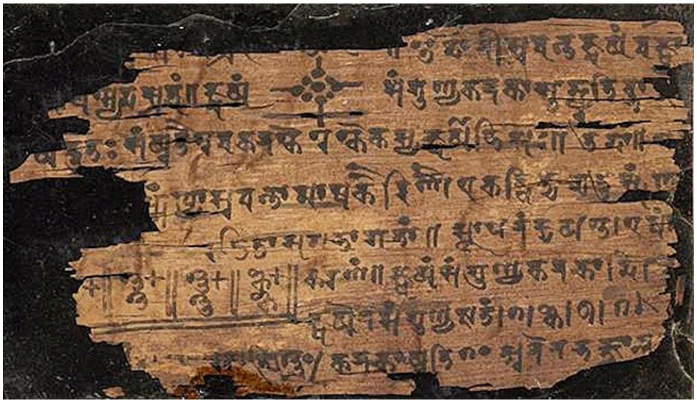


Figure 2: This manuscript is an early example of Hindu numerals.

Learning Objectives

After completing this section, you should be able to:

1. Evaluate an exponential expression.
2. Convert a Hindu-Arabic numeral to expanded form.
3. Convert a number in expanded form to a Hindu-Arabic numeral.

The modern system of counting and computing isn't necessarily natural. That different symbols are used to indicate different quantities or amounts is a relatively new invention. Simple marking by scratches or dots, one for each item being counted, was the norm long into human history. The modern system doesn't use repeated symbols to indicate more than one of a thing. It uses the place of a digit in a **numeral** to determine what that digit represents. A numeral is a symbol used to represent a number. A **number** is an abstract idea that represents quantity or amount.

Being clear about the difference between numeral and number is important. Just like a person can be called by various names, such as brother, father, husband, uncle, they are all representing the same person, John Smith. The person John Smith is the number, and the names brother, father, husband, and uncle are the numerals.

> **WHO KNEW?**
>
> *Hindu-Arabic Numerals*
>
> The numerals we currently use are referred to as Hindu-Arabic numerals, although they have changed as time has passed. Early forms of the numerals for 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 began in India, and passed through Persia to the Middle East. Place value was also employed in the early systems of India. Once this system was in north Africa and the Middle East, it spread to Europe, eventually replacing Roman numerals. Over time,

> the original symbols transformed into our modern ones. Read this article for another perspective on how the symbols began (based on the moon!).

The system we use for counting and computing uses place values based on powers of 10. In this section, we review exponents and our positional system.

Evaluating Exponential Expressions

Most modern numerical systems depend on place values, where the quantity represented depends not only on the digit, but also on where the digit is in the number. The place value is a power of some specific number, which means most numbering systems are actually exponential expressions. An **exponential expression** is any mathematical expression that includes exponents. So, evaluating such an expression means performing the calculation. In this chapter, we will be using exponents that are positive integer values. Before we do so, let's remind ourselves about exponents and what they represent. Suppose you want to multiply a number. Let's label that number a, by itself some number of times. Let's label the number of times b. We denote that as a^b. We say a, or the **base**, raised to the bth power, or the **exponent**. For example, if we are multiplying 13 by itself eight times, we write 13^8 and say 13 to the eighth power.

When computing exponential expressions, we should be careful to remember the order of operations. Using the order of operation rules, calculations inside the parentheses are done first, then exponents are calculated, then multiplication and division calculations are performed, and then addition and subtraction.

> **VIDEO**
> Exponential Notation

Example 1 **Evaluating an Exponential Expression**

Evaluate the following exponential expressions.

1. $4 \times 5^2 + 2 \times 6^3$
2. $6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0$
3. $3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0$

Solution

1. To evaluate, or calculate, this expression, we use order of operations, which means the exponents are done first, then multiplications, and then additions.

$$4 \times 5^2 + 2 \times 6^3 = 4 \times 5 \times 5 + 2 \times 6 \times 6 \times 6 = 4 \times 25 + 2 \times 216 = 100 + 432 = 532$$

2. To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, then the additions. Remember that any base raised to the exponent 0 is 1.

$$6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0 = 6 \times 8 \times 8 + 3 \times 8 + 4 \times 1 = 6 \times 64 + 3 \times 8 + 4 \times 1 = 384 + 24 + 4 = 412$$

3. To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, and then the additions. Remember that any base raised to the exponent 0 is 1.

$$3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0 = 3 \times 100 + 0 \times 10 + 6 \times 1 = 300 + 0 + 6 = 306$$

Converting Hindu-Arabic Numerals to Expanded Form

When you see the number 738, and you speak the number out loud, what do you say? You probably said "seven hundred thirty-eight" while wondering what point could possibly be made by asking this. What you didn't say was "seven, and three, and eight." A pre-K student might say that. Which should make you wonder, why?

The reason is that you've been taught **place values**, or the positions of digits in a number that determine the values of those digits. You know that in a three-digit number, the first digit is hundreds, the second digit is tens, and the last digit is ones. These place values rely on powers of 10, which makes this system a **base 10 system**.

This sense of place value is what makes our system of numbers so useful. You've also been taught the **Hindu-Arabic numeration system**. This system, which uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, and also employs place value based on powers of 10, is in use today.

Writing a number using these place values is writing them in **expanded form**. For a number with n digits, the expanded form is the first digit times 10 raised to one less than n, plus each following digit times 10 raised to one less than the previous power of 10. For example, the number 738 would be written as $7 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$.

What about a four-digit number, like 5,825? Out loud, we'd say five thousand, seven hundred twenty-five. In expanded form, it would be $5 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$. Notice that the largest exponent is one less than the number of digits, and that the exponents go down by one as we move through the number.

PEOPLE IN MATHEMATICS

Aryabhata of Kusumapura and Brahmagupta

The Hindu-Arabic numeral system developed in India, and Aryabhata of Kusumapura is credited with the place value notation in the 5th century. However, the system wasn't as complete as it could be, until. roughly a century later, when Brahmagupta introduced the symbol for 0. The 0 is necessary to indicate that a given place value has been skipped, as in 4,098. In 4,098, the 10^2 power is skipped. Without such a symbol, 4,098 and 498 look similar. The value of both the place value notation and the introduction of the symbol 0 cannot be overstated, for math and the sciences.

Example 2 **Writing a Number in Expanded Form**

Write the following in expanded form.

1. 563
2. 4,821
3. 903,786

Solution

1. **Step 1:** Since there are three digits in 563, n is 3. So, this is the first digit times 10 raised to the power of 2, so we start with 5×10^2.

Step 2: Then we add the next digit, 6, multiplied by 10 to a power one less than the previous, at which point we have $5 \times 10^2 + 6 \times 10^1$.

Step 3: Finally, the last digit is multiplied by 10 to the zeroth power and added to the previous. This results in $5 \times 10^2 + 6 \times 10^1 + 3 \times 10^0$.

1. **Step 1:** Since there are four digits in 4,821, n is 4. We multiply the first digit, 4, by 10 raised to the power of 3, which is 4×10^3.

Step 2: Then we add the next digit, 8, multiplied by 10 to a power one less than the previous, at which point we have $4 \times 10^3 + 8 \times 10^2$.

Step 3: We continue to the next digit, lowering the exponent of 10 by one. Now we have $4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1$.

Step 4: Finally, the last digit is multiplied by 10 to the zeroth power and added to the previous. This results in $4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$.

1. Since there are six digits in 903,786, n is 6. So, we begin the process with 9 times 10 raised to the 5th power and continue through the numbers, reducing the exponent of 10 by one each time. This results in $9 \times 10^5 + 0 \times 10^4 + 3 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 6 \times 10^0$.

Converting Numbers in Expanded Form to Hindu-Arabic Numerals

Converting from expanded form back into a Hindu-Arabic numeral is the reverse process of expanding a number, and is equivalent to evaluating the exponential expression.

Example 3 **Converting a Number from Expanded Form to a Hindu-Arabic Numeral**

Convert the following into Hindu-Arabic numerals.

1. $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0$
2. $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0$
3. $6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$

Solution

1. Evaluating the expression results in: $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0 = 3 \times 100 + 4 \times 10 + 8 \times 1 = 300 + 40 + 8 = 348$

2. Evaluating the expression results in: $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0 = 5 \times 1000 + 0 \times 100 + 9 \times 10 + 9 \times 1 = 5000 + 0 + 90 + 9 = 5099$
3. Evaluating the expression results in: $$\begin{aligned} & 6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 1 \\ &= 6 \times 1,000,000 + 2 \times 100,000 + 0 \times 10,000 + 9 \times 1,000 + 1 \times 1 \\ &= 6,000,000 + 200,000 + 0 + 9,000 + 100 + 10 + 7 \\ &= 6,209,117 \end{aligned}$$

Key Terms

- numeral
- number
- exponential expression
- base
- exponent
- place value
- base 10 system
- Hindu-Arabic numeration system
- expanded form

Key Concepts

- Exponents are used to represent repeated multiplication of a base.
- In arithmetic, exponents are computed before multiplication, division, addition, and subtraction. Computing an exponent is done by multiplying the base by itself the number of times equal to the exponent.
- The system of numbers currently used is the Hindu-Arabic system. Digits in this system take on values based on their place in the number. The place values are determined by multiplying the digit by 10 raised to the appropriate power.
- The expanded form of a Hindu-Arabic number is the sum of each digit times 10 raised to the exponent for that place value.

Videos

- Exponential Notation

4.2 Early Numeration Systems

Figure 3: Babylonians used clay tablets for writing and record keeping.

Learning Objectives

After completing this section, you should be able to:

1. Understand and convert Babylonian numerals to Hindu-Arabic numerals.
2. Understand and convert Mayan numerals to Hindu-Arabic numerals.
3. Understand and convert between Roman numerals and Hindu-Arabic numerals.

Each culture throughout history had to develop its own method of counting and recording quantity. The system used in Australia would necessarily differ from the system developed in Babylon that would, in turn, differ from the system developed in sub-Saharan Africa. These differences arose due to cultural differences. In nearly all societies, knowing the difference between one and two would be useful. But it might not be useful to know the difference between 145 and 167, as those quantities never had a practical use. For example, a shepherd likely didn't manage more than 100 sheep, so quantities larger than 100 might never have been encountered. This can even be seen in our use of the term few, which is an inexact quantity that most would agree means more than two. However, as societies became more complex, as commerce arose, as military bodies developed, so did the need for a system to handle large numbers. No matter the system, the issues of representing multiple values and how many symbols to use had to be addressed. In this section, we explore how the Babylonians, Mayans, and Romans addressed these issues.

Understand and Convert Babylonian Numerals to Hindu-Arabic Numerals

The Babylonians used a mix of an **additive system of numbers** and a **positional system of numbers**. An additive system is a number system where the value of repeated instances of a symbol is added the number of times the symbol appears. A positional system is a system of numbers that multiplies a "digit" by a number raised to a power, based on the position of the "digit."

The Babylonian place values didn't use powers of 10, but instead powers of 60. They didn't use 60 different symbols though. For the value 1, they used the following symbol:

For values up to 9, that symbol would be repeated, so three would be written as

To represent the quantity 10, they used

For 20, 30, 40, and 50, they repeated the symbol for 10 however many times it was needed, so 40 would be written

When they reached 60, they moved to the next place value. The complete list of the Babylonian numerals up to 59 is in .

You can see how Babylonians repeated the symbols to indicate multiples of a value. The number 6 is 6 of the symbol for 1 grouped together. The symbol for 30 is three of the symbols for 10 grouped together. However, their system doesn't go past 59. To go past 59, they used place values. As opposed to the Hindu-Arabic system, which was based on powers of 10, the Babylonian positional system was based on powers of 60. You should also notice there is no symbol for 0, which has some impact on the number system. Since the Babylonian number system lacked a 0, they didn't have a placeholder when a power of 60 was absent. Without a 0, 101, 110, and 11 all look the same. However, there is some evidence that the Babylonians left a small space between "digits" where we would use a 0, allowing them to represent the absence of that place value. To summarize, the **Babylonian system of numbers** used repeating a symbol to indicate more than one, used place values, and lacked a 0.

WHO KNEW?

Invention of 0

> The idea of 0 is not a natural one. Most cultures failed to recognize the need for a 0. If someone asked a farmer in 300 B.C.E. how many cows they had, but they had none, they would not answer “zero.” They’d say “I don’t have any” and be done with it. It wasn’t until roughly 3 B.C.E. that 0 appeared in Mesopotamia. It was independently discovered (or invented!) in the Mayan culture around 4 C.E. it made its appearance in India in the 400s C.E., and began to spread at that point. It wasn’t developed earlier mostly because positional systems were not yet fully developed. Once positional systems arose, the need to represent a missing power had to be addressed.

So how do we convert from Babylonian numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from , and then place values based on powers of 60. If you have n digits in the Babylonian number, you multiply the first “digit” by 60 raised to one less than the number of “digits.” You then continue through the “digits,” multiplying each by 60 raised to a power that is one smaller. However, be careful of spaces, since they represent a zero in that place.

Example 1 **Converting Two-Digit Babylonian Numbers to Hindu-Arabic Numbers**

Convert the Babylonian number

into a Hindu-Arabic number.

Solution

has two digits:

and

Step 1: So the first symbol,

represents 4 in the Babylonian system. This is multiplied by 60 to the first power (just as would happen in a two digit number), which gives us 4×60^1.

Step 2: The next symbol is

which represents 27 in the Babylonian system. This is multiplied by 60 raised to 0, which gives $4 \times 60^1 + 27 \times 60^0$.

Step 3: Calculating that yields $4 \times 60^1 + 27 \times 60^0 = 240 + 27 = 267$. So the Babylonian number

equals 267 in the Hindu-Arabic number system.

Example 2 **Converting Three-Digit Babylonian Numbers to Hindu-Arabic Numbers**

Convert the Babylonian number

into a Hindu-Arabic number.

Solution

has three digits:

and

and

Step 1: So the first symbol,

represents 13 in the Babylonian system. This is multiplied by 60 to the second power (since there are 3 digits), which gives us 13×60^2.

Step 2: The next symbol is

which represents 8 in the Babylonian system, is multiplied by 60 raised to the first power, which gives us $13 \times 60^2 + 8 \times 60^1$.

Step 3: The last digit is

representing 54, which is multiplied by 60 raised to 0, which gives $13 \times 60^2 + 8 \times 60^1 + 54 \times 60^0$.

Step 4: Calculating that yields $13 \times 60^2 + 8 \times 60^1 + 54 \times 60^0 = 13 \times 3,600 + 8 \times 60 + 54 \times 1 = 46,800 + 480 + 54 = 47,334$.

So, the Babylonian number

equals 47,334 in the Hindu-Arabic number system.

Example 3 **Converting Four-Digit Babylonian Numbers to Hindu-Arabic Numbers**

Convert the Babylonian number

into a Hindu-Arabic number.

Solution

It appears that

has three digits, but there is a space in between

and

Remember, the Babylonian system has no 0, it instead employs a space where we expect a zero. This means this is a four digit number.

Step 1: The first symbol,

represents 12 in the Babylonian system. This is multiplied by 60 to the third power since there are four digits, which gives us 12×60^3.

Step 2: The next symbol is a blank, which for us is a 0, representing 0×10^2, giving us $12 \times 60^3 + 0 \times 10^2$.

Step 3: The next symbol is

which represents 42 in the Babylonian system, is multiplied by 60 raised to the first power, which gives us $12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1$.

Step 4: The last Babylonian digit,

represents 39 in the Babylonian system. This is multiplied by 60 raised to 0, which gives $12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1 + 39 \times 60^0$.

Step 5: Calculating that yields

$$
\begin{aligned}
&12 \times 60^3 + 0 \times 10^2 + 42 \times 60^1 + 39 \times 60^0 \\
&= 12 \times 216,000 + 0 \times 10^2 + 42 \times 60 + 39 \times 1 \\
&= 2,592,000 + 0 + 2,520 + 39 \\
&= 2,594,559
\end{aligned}
$$

So the Babylonian number

equals 2,594,559 in the Hindu-Arabic number system.

WHO KNEW?

The Legacy of Babylonian System

The Babylonian system can still be seen today. An hour is 60 minutes, and a minute is 60 seconds. Additionally, when measuring angles in degrees, each degree can be split into 60 minutes (1/60th of a degree) and 60 seconds (1/60th of a minute).

VIDEO

Converting Between Babylonian and Hindu-Arabic Numbers

Understand and Convert Mayan Numerals to Hindu-Arabic Numerals

The Mayans employed a positional system just as we do and the Babylonians did, but they based their position values on powers of 20 and they had a dedicated symbol for zero. Similar to the Babylonians, the Mayans would repeat symbols to indicate certain values. A single dot was a 1, two dots were a 2, up to four dots. Then a five was a horizontal bar. The horizontal bars could be used three times, since the fourth horizontal bar would make a 20, which was a new position in the number. The 0 was a special picture, which appears like a turtle lying on its back. The shell would then be "empty," so maybe that's why the symbol was 0. The complete list is provided in . Another feature of Mayan numbers was that they were written vertically. The powers of 20 increased from bottom to top.

To summarize, the **Mayan system of numbers** used repeating symbol to indicate more than one, used place values, and employed a 0. So how do we convert from Mayan numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from and then place values based on powers of 20. If you have n digits in the Mayan number, you multiply the first "digit" by 20 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 20 raised to a power that is one smaller than the previous power. Fortunately, there is an explicit 0, so there is no ambiguity about numbers like 110, 101, and 11.

Example 4 **Converting Two-Digit Mayan Numbers to Hindu-Arabic Numbers**

Convert the Mayan number

into a Hindu-Arabic number.

Solution

has two digits:

and

Step 1: So, the first symbol,

represents 15 in the Mayan system. This is multiplied by 20 to the first power, which gives us 15×20^1.

Step 2: The next symbol is

which represents 9 in the Mayan system. This is multiplied by 20 raised to 0, which gives $15 \times 20^1 + 9 \times 20^0$.

Step 3: Calculating that yields $15 \times 20^1 + 9 \times 20^0 = 300 + 9 = 309$. So

equals 309 in the Hindu-Arabic number system.

Example 5 **Converting Three-Digit Mayan Numbers to Hindu-Arabic Numbers**

Convert the Mayan number

into a Hindu-Arabic number.

Solution

has three digits:

and

and

Step 1: So the first symbol,

represents 6 in the Mayan system. This is multiplied by 20 to the second power (since there are 3 digits), which gives us 6×20^2.

Step 2: The next symbol is

which represents 8 in the Mayan system, is multiplied by 20 raised to the first power, which gives us $6 \times 20^2 + 8 \times 20^1$.

Step 3: The last digit is

representing 4, which is multiplied by 20 raised to 0, which gives $6 \times 20^2 + 8 \times 20^1 + 4 \times 20^0$.

Step 4: Calculating that yields $6 \times 20^2 + 8 \times 20^1 + 4 \times 20^0 = 6 \times 400 + 8 \times 20 + 4 \times 1 = 2,400 + 160 + 4 = 2,564$. So the Mayan number

equals 2,564 in the Hindu-Arabic number system.

Example 6 **Converting Four-Digit Mayan Numbers to Hindu-Arabic Numbers**

Convert the Mayan number

into a Hindu-Arabic number.

Solution

has four digits, so the first power of 20 that is used is 3.

Step 1: The first symbol,

represents 8 in the Mayan system. This is multiplied by 20 to the third power (since there are four digits), which gives us 8×20^3.

Step 2: The next symbol is

which is a 0, representing 0×20^2, giving us $8 \times 20^3 + 0 \times 20^2$.

Step 3: The next symbol is

which represents 16 in the Mayan system, is multiplied by 20 raised to the first power, which gives us $8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1$.

Step 4: The last Mayan digit,

represents 5 in the Mayan system. This is multiplied by 20 raised to 0, which gives $8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1 + 5 \times 20^0$.

Step 5: Calculating that yields

$$\begin{aligned} &8 \times 20^3 + 0 \times 20^2 + 16 \times 20^1 + 5 \times 20^0 \\ &= 8 \times 8000 + 0 \times 400 + 16 \times 20 + 5 \times 1 \\ &= 64,000 + 0 + 320 + 5 \\ &= 64,325 \end{aligned}$$

So the Mayan number

equals 64,325 in the Hindu-Arabic number system.

WHO KNEW?

The Mayan Calendar

The Mayans used this base 20 system for everyday situations. But their culturally important, and extremely accurate, calendar system used a slightly different system. For their calendars, they used a system where the place values were 1, 20, then 20*18, then 20*18*18. The reason for this is 20*18 is 360, which is closer to the number of days in a year. Had they used a purely base 20 system for their calendar, they'd be very far off with 400 days in a year.

Three hundred sixty days still left the Mayans a bit short, as there are 365 days in a year (ignoring leap years). The Mayan calendar also included 5 days, called Wayeb days, which brings their calendar to 365 days. As it happens, Wayeb is the Mayan god of misfortune, so these 5 days were considered the bad luck days.

VIDEO

Converting Mayan Numbers to Hindu-Arabic Numbers

Understand and Convert Between Roman Numerals and Hindu-Arabic Numerals

The Mayan and Babylonian systems shared two features, one of which we are familiar with (place value) and one that we don't use (repeated symbols). The **Roman system of numbers** used repeated symbols, but does not employ a place value. It also lacks a 0. The Roman system is built on the following symbols in .

Roman Numeral	Hindu-Arabic Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1,000

As in the Mayan and Babylonian systems, a symbol may be repeated to indicate a larger value. However, at 4, they did not use IIII. They instead used IV. Since the I came before the V, the number stands for "one before five." A similar process was used for 9, which was written IX, or "one before ten." The value 40 was written XL, or "ten before fifty," while 49 was written XLIX, or "forty plus nine."

The following are the rules for writing and reading Roman numerals.

- The representations for bigger values precede those for smaller values.
- Up to three symbols may be grouped together; for example, III for 3, or XXX for 30, or CC for 200.
- A larger value followed by a smaller value indicated addition; for example, VII for 7, XIII for 13, LV for 55, and MCC for 1200.
- I can be placed before V to indicate 4, or before X, to indicate 9. These are the only ways I is used as a subtraction.
- X can be placed before L to indicate 40, and before C to indicate 90. These are the only ways X is used as a subtraction.
- C can be placed before D to indicate 400, and before M to indicate 900. These are the only ways C is used as a subtraction.
- If multiple symbols are used, and a subtraction involving that symbol, the subtraction part comes after the multiple symbols. For example, XXIX for 29 and CCXC for 290.

WHO KNEW?

Legacy of Roman Numerals

The Roman numbering system is still used today in some situations. Many cornerstones of buildings have the year written in Roman numerals. Movie titles often represent the year the movie was produced as Roman numerals. The most recognizable might be that the Super Bowl is numbered using Roman numerals.

Example 7 **Converting Roman Numerals to Hindu-Arabic Numbers**

Convert the following Roman numerals into Hindu-Arabic numerals.

1. XXVII
2. XXXIV
3. MMCMXLVIII

Solution

1. The numeral XXVII begins with two X's, which is then followed by a V. So, the two X's combine to be 20. The V is followed by two I's, so the V indicates the addition of 5. The two I's that follow indicate addition of two. That ends the symbols, so the value is 20 plus 5 plus 2, or 27 in Hindu-Arabic numerals.
2. The numeral XXXIV begins with three X's, which is then followed by an I. So, the three X's combine to be 30. The I is followed by a V, which indicates 4. That ends the symbols, so the value is 30 plus 4, or 34 in Hindu-Arabic numerals.

3. The numeral MMCMXLVIII begins with two M's, which is then followed by a C. So, the two M's combine to make 2000. The C is followed by an M, which indicates 900. The CM is followed by XL, which indicates 40. The L is followed by V, which indicates 5. The V is followed by three I's, indicating 3. Adding those values yields 2,948.

VIDEO

Converting From Roman Numbers to Hindu-Arabic Numbers

Of course, we can convert from Hindu-Arabic numerals, to Roman numerals, too.

Example 8 **Converting Hindu-Arabic Numbers to Roman Numerals**

Convert the following Hindu-Arabic numerals into Roman numerals.

1. 38
2. 94
3. 846
4. 2,987

Solution

1. Thirty is represented as three X's, and the 8 is represented with VIII, so 38 in Roman numerals is XXXVIII.
2. Ninety is represented by XC, and four is represented by IV, so 94 in Roman numerals is XCIV.
3. The number is less than 900 and more than 500, so the first symbol to be used is D, which is 500. To get to 800, we need 300 more, which is represented with three C's. Forty is represented with XL, and the six. The Roman numerals are DCCCXLVI.
4. The two thousand is represented by two M's. The 900 is represented by CM. The 80 is represented by LXXX (50 plus 30). Finally, the 7 is represented by VII. We have that 2,987 in Roman numerals is MMCMLXXXVII.

VIDEO

Converting From Hindu-Arabic Numbers to Roman Numbers

Key Terms

- additive system of numbers
- positional system of numbers
- Babylonian system of numbers
- Mayan system of numbers

- Roman system of numbers

Key Concepts

- Historically, there have been many systems for numbering. One system is an additive system, in which symbols are repeated to express larger numbers. Another system is a positional system, in which the digits and their positions determine the quantity being represented.
- The Babylonian system was a combination of a positional and additive system. It used 60 as its base. Using that in the positional system makes it possible to convert between Babylonian and Hindu-Arabic numbers.
- The Mayan system was a combination of a positional and additive system. It used 20 as its base. Using that in the positional system makes it possible to convert between Mayan and Hindu-Arabic numbers.
- The Roman system was an additive system. Knowing what each symbol represents makes it possible to convert between Roman and Hindu-Arabic numbers.

Videos

- Converting Between Babylonian and Hindu-Arabic numbers
- Converting Mayan Numbers to Hindu-Arabic Numbers
- Converting From Roman Numbers to Hindu-Arabic Numbers
- Converting From Hindu-Arabic Numbers to Roman Numbers

4.3 Converting with Base Systems

Figure 4: Computers use Base 2, which only uses 0′s and 1′s, to represent quantity.

Learning Objectives

After completing this section, you should be able to:

1. Convert another base to base 10.
2. Write numbers in different base systems.
3. Convert base 10 to other bases.

4. Determine errors in converting between bases.

   In our system of numbers, we use base 10, but using base 10 was not a given within other systems. There were other systems that used bases other than 10, as we saw with the Mayans and the Babylonians. The base 10 system comes down to grouping objects in sets of 10, but grouping in sets of 10 only happens if the culture values grouping by that many. We feel 10 is natural because we have 10 fingers. There are other systems using other grouping values, such as 4 or 20.

   One good reason for examining other bases is to remind ourselves how we had to learn arithmetic when we were young, memorizing rules for our base 10 system. We had to learn why those arithmetic rules made sense, such as why $1 + 1 = 2$ and $1 + 2 = 3$. Another good reason for learning other base systems is due to computers; their circuitry instead uses base 2.

   In this section, we explore other base systems and how to convert between them.

Conversion of Another Base into Base 10 and Other Bases

We saw in Hindu-Arabic Positional System that our Hindu-Arabic system uses **base 10**, which is a system using place values of digits that depend on powers of 10 (or, are based on powers of 10). We've already worked with bases other than base 10: The Babylonian system was base 60, while the Mayan system was base 20.

To explore how our base 10 system is used, answer the following question: What's the following quantity: 4,572? You probably said four thousand five hundred seventy-two (no, there is no "and" between hundred and seventy). But why do you think that 4 means four thousand? A very young person when learning their numbers might say that's a four five seven and two. But you added the context of thousands to the four. Why?

Place value, that's why. You learned early on that where the numeral was gave it different meanings. Ten thousands, thousands, hundreds, tens, and ones. So, you translate that symbol string (4,572) into "four thousand five hundred seventy-two." As we saw in Hindu-Arabic Positional System, expanding a Hindu-Arabic number involved writing the number using each digit times its appropriate power of 10. So, we could write 4,572 as $4 \times 10^3 + 5 \times 10^2 + 7 \times 10^1 + 2 \times 10^0 = 4 \times 1000 + 5 \times 100 + 7 \times 10 + 2 \times 1$.

One possible reason we use base 10 is that we have 10 fingers, and in the cultures where the Hindu-Arabic system developed, that became the standard. Other cultures may have used other ways of organizing numbers, perhaps using 20 by including toes, or using 60 because 60 has many divisors. Mathematically though, base 10 is an awkward base to work in since 10 has limited divisors. But we think it is easy and simple because that's what we've been taught to use.

Using a base 10 system means we need 10 symbols to make our numbering system work: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Now imagine that we all only had 6 fingers instead of 10 and our counting system was based on those 6 fingers. We would be counting in groups of 6, not groups of 10. How would this change how we work with quantity?

First, we'd need only six symbols. Let's use 0, 1, 2, 3, 4, 5. Second, our place values would be based on powers of 6, not powers of 10. For instance, the number 3,024 in base 6 would be

$3 \times 6^3 + 0 \times 6^2 + 2 \times 6^1 + 4 \times 6^0$. That is how you can translate a base 6 number into a base 10 number. When we calculate that expression we get $3 \times 6^3 + 0 \times 6^2 + 2 \times 6^1 + 4 \times 6^0 = 3 \times 216 + 0 \times 36 + 2 \times 6 + 4 \times 1 = 648 + 0 + 12 + 4 = 664$.

This means the base 6 number 3,024 is equal to the base 10 number 664.

From now on, if we are using a base 6 number, we will follow it with the subscript 6, like the following: $3{,}024_6$ means the number is in base 6.

A base 10 number gets no subscript (it's the standard). So, 3,024 is a base 10 number. A base 13 number would be $4{,}672_{13}$.

So, a base 6 system uses only the symbols 0, 1, 2, 3, 4, and 5. Also, the place values use powers of 6. However, we still don't know how to count in base 6. In order to do so, we'd have to know how to represent the quantities larger than five in base 6. Let's review how our base 10 system works by counting from 0 to 100, which shows how larger values are represented.

In writing the base 10 numbers, you start with these first 10 values:

0	1	2	3	4	5	6	7	8	9

But you've run out of symbols. So, we use two digits:

10	11	12	13	14	15	16	17	18	19

The 1 out front means you've run out of digits one time.

But now you've run out twice. Continuing with those numbers gives:

20	21	22	23	24	25	26	27	28	29

And so on,

30	31	32	33	34	etc.…

Eventually, you hit the 90s,

90	91	92	93	94	95	96	97	98	99

And you've run out of the digits again! So, we say we've run out of digits in the tens place one time, hence:

100	101	102	103	104	105	106	107	108	109

That's the pattern we use in base 10. We write out the symbols until we've used all the symbols, then add a digit in front that counts how many times we've used the digits. Knowing the numbers, or being able to count higher and higher, is necessary to understand how all the arithmetic works, as it all goes back to counting.

The counting pattern is the same for any other base, including base 6. So, let's start:

0	1	2	3	4	5

But we've run out of symbols! Just like in base 10, we use a second digit, where the first digit will tell us we've run out of symbols one time.

10	11	12	13	14	15

And we use the same pattern:

20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

But we've run out of symbols for that front digit. So, we indicate it the same way as in base 10… by adding a third digit in front, indicating we've run out of symbols once in the second place:

100	101	102	103	104	105
110	111	etc.			

The symbol pattern is the same, but truncated. We only use the six symbols. So that is how we represent base 6. Being able to write out these numbers is important when working with addition in the base.

When using a base larger than 10, though, we need more symbols. Instead of creating new symbols, we use capital letters, with A representing the digit for "10," B representing the digit for "11," and so on.

Example 1 Determining Digits of a Base with Less Than 10 Digits

What are the digits used for base 7?

Solution

Since this is base 7, we need only 7 symbols: 0, 1, 2, 3, 4, 5, 6.

Example 2 Determining Digits of a Base with More Than 10 Digits

What are the digits used for base 14?

Solution

Since this is base 14, we need 14 symbols. We don't have single character numbers for 10, 11, 12, and 13, so, in a fit of inspired creativity, we use capital letters A, B, C, D to represent those quantities. So, the digits in base 14 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D.

WHO KNEW?

Using Base 12

As mentioned in the text, working in base 10 is mathematically awkward. Ten has only two natural number divisors: 2 and 5. This means dividing into groups is not easy. However, 12, or a dozen, has more divisors: 2, 3, 4, and 6. The Dozenal Society recognizes this more mathematically pleasant detail. It advocates for a switch to using base 12 for numbers. Their argument is based on the divisibility of the number 12. But has there ever been a society that used such a system? The answer is yes. A dialect of the Gwandara language in Nigeria uses the base 12 system. It is unlikely, though, that the Dozenal Society will achieve their goal, as the base 10 system is so entrenched in our society.

Example 3 **Converting from One Base into Another**

Convert $3{,}601_7$ into base 10.

Solution

In base 7, the place values are powers of 7. Since there are four digits, the highest power of 7 that is used is 3. This yields $3,601_7 = 3 \times 7^3 + 6 \times 7^2 + 0 \times 7^1 + 1 \times 7^0 = 3 \times 343 + 6 \times 49 + 0 \times 7 + 1 = 1,029 + 294 + 0 + 1 = 1,324$.

VIDEO

Convert Base 7 to Base 10

Example 4 **Converting from Base 14 to Base 10**

Convert $4B7_{14}$ into base 10.

Solution

In base 14, the place values are powers of 14. Since there are three digits, the highest power of 14 is 2. Also recall that in base 14, 10 is represented by A, 11 is represented by B, 12 is represented by C, and 13 is represented by D. Using that, we convert to base 10:

$$4B7_{14} = 4 \times 14^2 + B \times 14^1 + 7 \times 14^0 = 4 \times 196 + 11 \times 14 + 7 \times 1 = 784 + 154 + 7 = 945$$

Example 5 **Converting from Base 12 to Base 10**

Convert $A16_{12}$ into base 10.

Solution

In base 12, the place values are powers of 12. Since there are three digits, the highest power of 12 is 2. Also recall that in base 12, 10 is represented by A and 11 is represented by B. Using that, we convert to base 10:

$$A16_{14} = 10 \times 12^2 + 1 \times 12^1 + 6 \times 12^0 = 10 \times 144 + 1 \times 12 + 6 \times 1 = 1,440 + 12 + 6 = 1,458$$

Example 6 **Converting from Base 2 to Base 10**

Convert 1011_2 into base 10.

Solution

In base 2, the place values are powers of 2. Since there are four digits, the highest power of 2 is 3. Using that, we convert to base 10:

$$1011_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 1 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1 = 8 + 0 + 2 + 1 = 11$$

WHO KNEW?

Before Napoleon

Before Napoleon's France, which adopted the base 10 system, a modified base 12 system was often used in Europe. Twelve is easily divisible into groups of 2, 3, 4, and 6, which makes it easier to work with. Even our numbering system retains a bit of this. You have likely noticed that we use the words *thirteen*, *fourteen*, *fifteen*, and so on to indicate 10 and 3, 10 and 4, 10 and 5, and so one. Even the 20s reinforce this idea, as in twenty-one, and twenty-two. However, two numbers don't follow this pattern, namely 11 and 12. If they followed the same rules, they'd be one teen and two teen. We even have a special word for 12; that is, a dozen. However, etymologically speaking, the words *eleven* and *twelve* are likely derived by referencing the number 10. These two numbers may date back to the Old English words *endleofan* and *twelf*, which can be traced back further to *ain lif* and *twa lif*. The word *lif* here may be the base word for "to leave." This would suggest *ain lif* is one left after 10, and *twa lif* is two left after 10, or, 11 and 12.

Example 7 **Writing Numbers in Base Systems Other Than Base 10**

Write the numbers in base 7 up to 100_7.

Solution

Step 1: Using the patterns we indicated earlier, we begin with the first seven digits.

0, 1, 2, 3, 4, 5, 6

Step 2: Since we've run out of digits, we start with 10, indicating we've run out of symbols once.

10, 11, 12, 13, 14, 15, 16

Step 3: Continuing in the same way, we get:

20, 21, 22, 23, 24, 25, 26

30, 31, 32, 33, 34, 35, 36

40, 41, 42, 43, 44, 45, 46

50, 51, 52, 53, 54, 55, 56

60, 61, 62, 63, 64, 65, 66

Now, all the digits have been used in the leading digits. Since the digits have all been used in that leading digit, we use 100, as in base 10.

100

Example 8 **Writing Numbers in Bases with More Than 10 Symbols**

Write the numbers in base 14 up to 100_{14}.

Solution

Step 1: Using the patterns we indicated earlier, we begin with the first 14 digits.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D

Step 2: Since we've run out of digits, we start with 10, indicating we've run out of symbols once.

10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D

Step 3: Continuing in the same way, we get:

20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2A, 2B, 2C, 2D

30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 3A, 3B, 3C, 3D

40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 4A, 4B, 4C, 4D

50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 5A, 5B, 5C, 5D

60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 6A, 6B, 6C, 6D

70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 7A, 7B, 7C, 7D

80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 8A, 8B, 8C, 8D

90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 9A, 9B, 9C, 9D

A0, A1, A2, A3, A4, A5, A6, A7, A8, A9, AA, AB, AC, AD

B0, B1, B2, B3, B4, B5, B6, B7, B8, B9, BA, BB, BC, BD

C0, C1, C2, C3, C4, C5, C6, C7, C8, C9, CA, CB, CC, CD

D0, D1, D2, D3, D4, D5, D6, D7, D8, D9, DA, DB, DC, DD

100

Base 2 is important in the digital age, as it is the system used by computers. It is the simplest base to work with, but has the drawback that the numbers in base 2 may use many, many digits. In Addition and Subtraction in Base Systems and Multiplication and Division in Base Systems, we will look at base 2 in each situation.

Example 9 **Writing Numbers in Base 2**

Write the numbers in base 2 up to 100_2.

Solution

Base 2 uses only two symbols: 0 and 1. Following the pattern established previously, the numbers in base 2 up to 100_2 are 0, 1, 10, 11, and 100.

WHO KNEW?

Early Hawaiian Numeration System

Before the British arrived in Hawaii, people there used a system that combined two different bases. Objects were initially grouped into collections of four, and a collection of four was referred to as *kauna*. A person could have two *kauna* and three "ones" (in Hindu-Arabic, 11). Or they could have eight *kauna* and one "ones" (In Hindu-Arabic, 33). However, sets of four were grouped in collections of 10. A set of 10 *kauna* was *ka'au*. The collections of *ka'au* were grouped by 10 also. Which meant that 10 *ka'au* (this is 40 in Hindu-Arabic) would be *lau* (or 400 in Hindu-Arabic). What this shows is that the Hawaiian culture developed a system that used base 4 combined with base 10.

Conversion of Base 10 into Another Base

Converting from base 10 into another base uses repeated division, recording the **remainder** at each step. Then, the number in the new base is the remainder starting from the last remainder found. To be accurate in what we're saying, we need to remind ourselves of some terminology associated with division. When integers are divided, the one being divided is the **dividend**, and the one that is dividing the dividend is the **divisor**. The **quotient** is the largest natural number that can be multiplied by the divisor where the product is less than the dividend.

When the integer n is divided by the integer d, n is called the **dividend** and d is the **divisor.**

To convert a base 10 number n into base d, we divide n by d, recording the remainder. Then we divide the quotient from that step by the base d, and record the remainder again. We continue this process until the quotient is 0. Then, the base d number has digits that start with the last remainder and use each remainder in reverse order.

Example 10 **Converting from Base 10 into a Lower Base**

Convert 298 to base 6.

Solution

We divide 298 by 6, and record the remainder. Then we divide the quotient from that step by 6, and record the remainder again. We continue this process until the quotient is 0. Then, the base 6 number has digits that start with the last remainder and use each remainder in reverse order.

Step 1: When we divide 298 by 6, we get $6\,\overset{49r4}{\overline{)\,298}}$. The quotient is 49 and the remainder is 4.

Step 2: Now we divide the quotient, 49, by 6. This gives $6\,\overset{8r1}{\overline{)\,49}}$. The quotient is 8 and the remainder is 1.

Step 3: Repeating, we get $6\,\overset{1r2}{\overline{)\,8}}$. The quotient is 1 and the remainder is 2.

Step 4: Finally, we perform the operation on the quotient 1, $6\,\overset{0r1}{\overline{)\,1}}$ giving us a quotient of 0 and a remainder of 1.

Step 5: The base 6 number has digits equal to the remainders in reverse order, 1214_6. So, 298 in base 10 when converted to base 6 is 1214_6.

VIDEO

Converting from Base 10 to Another Base

Example 11 **Converting from Base 10 into a Higher Base**

Convert 45,134 to base 13.

Solution

We divide 45,134 by 13, and record the remainder. Then we divide the quotient from that step by 13, and record the remainder again. We continue this process until the quotient is 0. Then, the base 13 number has digits that start with the last remainder and use each remainder in reverse order. It is at this step that we'll convert to the base 13 digits, which are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C.

Step 1: When we divide 45,134 by 13, we get $13\,\overset{3,471r11}{\overline{)\,45{,}134}}$. The quotient is 3,471 and the remainder is 11.

Step 2: Now we divide the quotient, 3,471, by 13. This gives $13\,\overset{267r0}{\overline{)\,3{,}471}}$. The quotient is 267 and the remainder is 0.

Step 3: Repeating, we get $13\,\overset{20r7}{\overline{)\,267}}$. The quotient is 20 and the remainder is 7.

Step 4: Again, and we get $13\,\overset{1r7}{\overline{)\,20}}$. The quotient is 1 and the remainder is 7.

Step 5: Finally, we get $13\,\overset{0r1}{\overline{)\,1}}$, with quotient 0 and a remainder 1.

Step 6: The base 13 number has digits equal to the remainders in reverse order, which were 1, 7, 7, 0, and 11. The 11 is written as B in base 13. So, 45,134 in base 10 when converted to base 13 is 1770B_{13}.

Example 12 **Converting from Base 10 into Base 2**

Convert 100 to base 2.

Solution

Following the pattern above:

Step 1: We divide 100 by 2, and record the remainder.

Step 2: Then we divide the quotient from that step by 2, and record the remainder again.

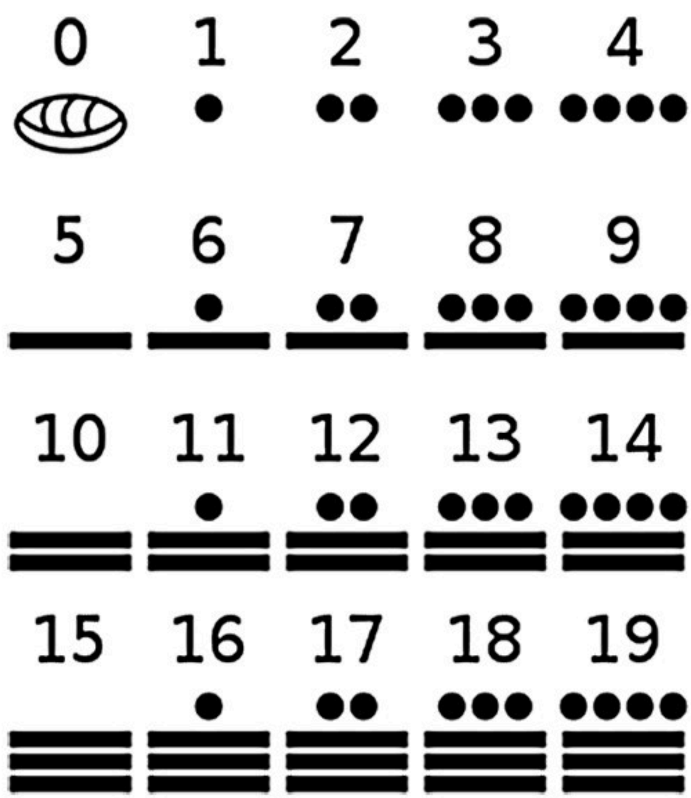
Step 3: We continue this process until the quotient is 0.

Step 4: Following this process, the remainders are, in order, 0, 0, 1, 0, 0, 1, 1. Writing those in reverse order gives the number in base 2, 1100100_2.

Notice that 100 in base 2 took seven digits.

Converting from Hindu-Arabic Numbers to Mayan Numbers

To convert from a Hindu-Arabic number to a Mayan number involves two distinct processes. First, the number must be converted to base 20, using the process described and demonstrated previously. Next, that base 20 number has to be written using Mayan numerals. For reference, the Mayan numerals and their values are below.



Example 13 **Converting from Base 10 into the Mayan System**

Convert the following into Mayan numbers.

1. 51
2. 653

Solution

1. The Mayan system is base 20, so we must use 20 in the process from above. The first division has a quotient of 2 and remainder of 11. The 11 serves as the "ones" digit. Dividing that quotient, 2, by 20 has a quotient of 0 with a remainder of 2. The 2 becomes the "twenties" digit of the number. So, in base 20, the number would be 2 followed by 11. The Mayan symbols for 2 and 11 are and . Writing these vertically, with the "ones" digit on top, as appropriate for Mayan numbers, results in:

2. The Mayan system is base 20, so we must use 20 in the process from above. The first division, 673 divided by 20, has a quotient of 32 and remainder of 13. Dividing that quotient, 32, by 20 has a quotient of 1 with a remainder of 12. Dividing that quotient, 1, by 20 has a quotient of 0 and a remainder of 1. Since there are three remainders here, this is a three-digit number. The 1 is the "20-squared" digit, the 12 is the "twenties" digit, and the 13 is the "ones" digit. So, in base 20, the number would be 1 followed by 12 followed by 13.. The Mayan symbols for 1, 12 and 13 are , , and . Writing these vertically, as appropriate for Mayan numbers, would result in:

WHO KNEW?

Other Languages, Other Bases

There have been base systems that use bases other than 10. Some bases used were 20, 12, and 27! Visit this site to see more on the languages that used other bases.

Errors in Converting Between Bases

There are some common errors that are made when converting between bases. Often, it comes down to using an "illegal" symbol in the new base.

Example 14 **Detecting an Illegal Symbol When Converting Between Bases**

A base 10 number is converted to base 7 and the result was 2081_7. Was an error committed? How do you know?

Solution

The result has the digit 8 in it. In base 7, 8 is an illegal symbol. Based on that, an error was committed.

When converting from base 10 to another base, an illegal symbol will be used if a mistake was made in the division process used to find the number in the new base. Since the digits are based on the remainders, any remainder that is an illegal symbol would indicate an error.

Example 15 **Detecting an Error in Division When Converting Between Bases**

When changing from base 10 to base 8, the division process resulted in the following remainders: 1, 0, 9, 2, 4. Was an error committed? How do you know?

Solution

The remainders include 9, which in base 8 is an illegal symbol.

Another possible way to detect an error in converting between bases is to count the number of digits. When converting from a higher base to a lower base, the number of digits cannot get smaller. Similarly, when converting from a lower base to a higher base, the number of digits cannot get bigger. So, if a base 10 number is converted to a base 3 number, the number of digits in the new base 3 numbers cannot be less than the number of digits in the base 10 number. Similarly, if a base 7 number is converted to base 10, the number of digits in the base 10 number cannot be more than the number of digits in the original base 7 number.

Example 16 **Detecting an Error in Number of Digits When Converting Between Bases**

A five-digit base 10 number is converted to a base 5 number. The base 5 number has four digits. Was an error committed? How do you know?

Solution

Since 10 is larger than 5, the base 5 number cannot have less digits than the base 10 number. Since it did, we know an error has been made.

The Babylonian system used base 60. To convert from Hindu Arabic numbers into Babylonian numbers, the process for converting from base 10 to a different base would be done first. Then, the results found in the conversion process would be changed to Babylonian numerals. This process is similar to the one for Mayan numbers.

Key Terms

- base 10
- remainder
- dividend
- divisor
- quotient

Key Concepts

- The system we use is the base 10 system. Base 10 is not the only base that can be used. To use another base, one could start with a list of numbers in that base.
- To indicate that a number is written in a base other than 10, a subscript is appended to the end of the number. That subscript indicates the base for the number.
- Numbers written in a base smaller than 10 use the same symbols as base 10. However, when using bases larger than 10, the symbols A, B, C, … are used to represent digits larger than 9.

- To convert from a number written in a base other than 10 into a base 10 number, the number is written in expanded form and then that expression is computed.
- To convert a number from base 10 into another base, the base 10 number is repeatedly divided by the new base. The remainders when performing these divisions become the digits for the number in the new base.
- Common errors can be detected when performing base conversions.

Videos

- Convert Base 7 to Base 10
- Converting from Base 10 to Another Base

4.4 Addition and Subtraction in Base Systems

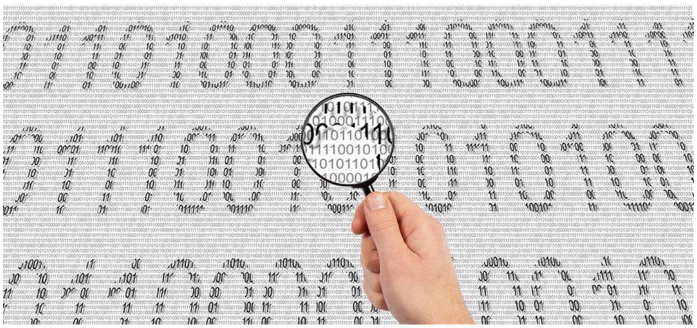


Figure 6: All information in computers is represented by 0′s and 1′s, including quantity, which means computers use Base 2 for arithmetic.

Learning Objectives

After completing this section, you should be able to:

1. Add and subtract in bases 2–9 and 12.
2. Identify errors in adding and subtracting in bases 2–9 and 12.

Once we decide on a system for counting, we need to establish rules for combining the numbers we're using. This begins with the rules for addition and subtraction. We are familiar with base 10 arithmetic, such as $2 + 5 = 7$ or $3 \times 5 = 15$. How does that change if we instead use a different base? A larger base? A smaller one? In particular, computers use base 2 for all number representation. When your calculator adds or subtracts, multiplies or divides, it uses base 2. This is because the circuitry recognizes only two things, high current and low current, which means the system is uses only has two symbols. Which is what base 2 is.

In this section, we use addition and subtraction in bases other than 10 by referencing the processes of base 10, but applied to a new base system.

Addition in Bases Other Than Base 10

Now that we understand what it means for numbers to be expressed in a base other than 10, we can look at arithmetic using other bases, starting with addition. When you think back to when you first learned addition, it is very likely you learned the addition table. Once you knew the addition table, you moved on to addition of numbers with more than one digit. The same process holds for addition in other bases. We begin with an addition table, and then move on to adding numbers with two or more digits.

We worked with base 6 earlier, and have the numbers in base 6 up to 100_6. Using that table of values, we can create the base 6 addition table.

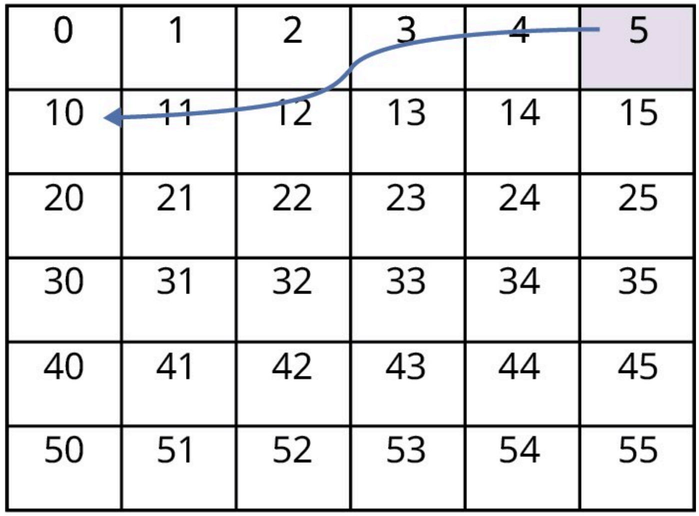
Here's the beginning of the base 6 addition table:

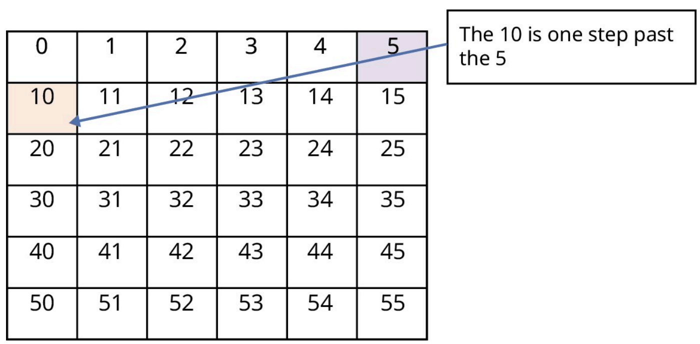
+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	?
2	2	3	4	5	?	?
3	3	4	5	?	?	?
4	4	5	?	?	?	?
5	5	?	?	?	?	?

Many of the cells are not filled out. The ones filled in are values that never get past 5, which is the largest legal symbol in base 6, so they are acceptable symbols. But what do we do with 5 + 3 in base 6? We can't represent the answer as "8" since "8" is not a symbol available to us. Let's go back to the list of numbers we have for base 6.

0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

So, what is $5 + 1$ equal to in base 6? Well, start at the 5, and jump ahead one step. You land on 10.





This means that, in base 6, $5 + 1 = 10$.

So, what is $5 + 2$ in base 6? Well, $5 + 2 = 5 + 1 + 1$, so $10 + 1$…jump one more space and you land on 11. So, $5 + 2 = 11$ in base 6.

0	1	2	3	4	5
10	11	12	13	14	15
20	21	22	23	24	25
30	31	32	33	34	35
40	41	42	43	44	45
50	51	52	53	54	55

And so it goes. Using that process, stepping one more along the list, we can fill in the remainder of the base 6 addition table .

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	**10**
2	2	3	4	5	**10**	**11**
3	3	4	5	**10**	**11**	**12**
4	4	5	**10**	**11**	**12**	**13**
5	5	**10**	**11**	**12**	**13**	**14**

With this table, and with our understanding of “carrying the one,” we can then use the addition table to do addition in base 6 for numbers with two or more digits, using the same processes you learned for addition when you did it by hand.

Example 1 **Adding in Base 6**

Calculate $251_6 + 133_6$.

Solution

Step 1: Let’s set up the addition using columns.

	2	5	1
+	1	3	3

Step 2: Let’s do the one’s place first. According to the base 6 addition table , $1 + 3 = 4$.

	2	5	1
+	1	3	3
			4

Step 3: Now, we do the “tens” place (it’s really the sixes place). According to the base 6 addition table , we have 5 + 3 = 12. So, like in base 10, we use the 2 and carry the 1.

	1		
	2	5	1
+	1	3	3
		2	4

Step 4: Now the “hundreds” place (really, thirty-sixes place). There, we have 1 + 2 + 1 = 3 + 1 = 4.

	1		
	2	5	1
+	1	3	3
	4	2	4

So, $251_6 + 133_6 = 424_6$.

As you can see, the process is the same as when you learned base 10 addition, just a different symbol set.

Example 2 **Creating an Addition Table for a Base Lower Than 10**

1. Create the addition table for base 7.
2. Create the addition table for base 2.

Solution

1. We begin with the table below.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	
2	2	3	4	5	6		
3	3	4	5	6			
4	4	5	6				
5	5	6					
6	6						

In base 7, the number that follows 6 is 10 (since we've run out of symbols!). So, $6_7 + 1_7 = 10_7$. Once that is established, $6_7 + 2_7$ will be two numbers past 6, which is 11 in base 7.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	**10**
2	2	3	4	5	6		**11**
3	3	4	5	6			
4	4	5	6				
5	5	6					
6	6	**10**	**11**				

Continuing, we can fill in the rows as we would in base 10, but being aware that we are working in base 7 .

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

1. We revisit base 2 here. Begin with the table:

+	0	1
0	0	1
1	1	

In base 2, the number that follows 1 is 10 (since we've run out of symbols!). So, $1_2 + 1_2 = 10_2$. The complete table for base two then is below.

+	0	1
0	0	1
1	1	10

This demonstrates that the rules necessary for base 2 addition are as small as possible: four rules.

To summarize the creation of the addition tables for a given base, do the following.

Step 1: Set up the table.

Step 2: Fill in all the additions that use the "legal" symbols for the base. The diagonal that goes from upper left to lower right that is immediately next to the filled boxes all get the value 10, regardless of base.

Step 3: Enter the values that are in the "teens." This can all be done on one table without creating multiple copies of previously done work.

Example 3 **Adding in Base 7**

Calculate $536_7 + 433_7$.

Solution

Step 1: Let's set up the addition using columns.

	5	3	6
+	4	3	3

Step 2: Let's do the one's place first. According to the base 7 addition table in the solution for Example 2, 6 + 3 = 12. We will carry the 1.

		1	
	5	3	6
+	4	3	3
			2

Step 3: Now, we do the “tens” place (it’s really the sevens place). According to the base 7 addition table in the solution for Example 2, we have 1 + 3 + 3 = 10. So, like in base 10, we use the 0 and carry the 1.

	1		
	5	3	6
+	4	3	3
		0	2

Step 4: Now the “hundreds” place (really, forty-ninths place). There, we have 1 + 5 + 4 = 6 + 4 = 13.

	1		
	5	3	6
+	4	3	3
1	**3**	0	2

So, $536_7 + 333_7 = 1302_7$.

As seen previously, when performing addition in another base, set up the problem exactly as you would for addition in base 10. At each step, check the addition table for the base. As in base 10 addition, move right to left, adding down the columns using the rules in the addition table. When necessary and just as in base 10, be sure to carry the 1.

Example 4 **Creating an Addition Table for a Base Higher Than 10**

Create the addition table for base 12.

Solution

Step 1: Recall, in base 12, the symbol set is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, and B. So, the addition table begins as shown below.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	
2	2	3	4	5	6	7	8	9	A	B		
3	3	4	5	6	7	8	9	A	B			
4	4	5	6	7	8	9	A	B				
5	5	6	7	8	9	A	B					
6	6	7	8	9	A	B						
7	7	8	9	A	B							
8	8	9	A	B								
9	9	A	B									
A	A	B										
B	B											

Step 2: The diagonal immediately to the right of the filled in boxes is where the 10 goes for this base.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	
3	3	4	5	6	7	8	9	A	B	10		
4	4	5	6	7	8	9	A	B	10			
5	5	6	7	8	9	A	B	10				
6	6	7	8	9	A	B	10					
7	7	8	9	A	B	10						
8	8	9	A	B	10							
9	9	A	B	10								
A	A	B	10									
B	B	10										

Step 3: Using the pattern we're familiar with, and counting in base 12, we can fill in the other cells.

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A	A	B	10	11	12	13	14	15	16	17	18	19
B	B	10	11	12	13	14	15	16	17	18	19	1A

Notice that the lower-right entry is $1A_{12}$, as this is the number one past 19_{12}.

Example 5 **Adding in Base 12**

Calculate $3A7_{12} + 9BA_{12}$.

Solution

Step 1: Using the process established in the earlier addition problem, set up the columns.

	3	A	7
+	9	B	A

Step 2: Using the rules from the base 12 addition table in the solution for Example 4, and being careful to carry the 1 when necessary, we get the following:

	1	1	
	3	A	7
+	9	B	A
1	**1**	A	5

The ones that were carried are located over the columns.

So, $3A7_{12} + 9BA_{12} = 11A5_{12}$.

Example 6 **Adding in Base 2**

We again return to base 2, the base used by computers. Calculate $1001_2 + 11011_2$.

Solution

Step 1: Using the process established in the earlier addition problem, set up the columns.

		1	0	0	1
+	1	1	0	1	1

Step 2: Using the rules from the base 2 addition table in the solution for Example 2, and being careful to carry the 1 when necessary (and shown at the top of the grid), we get the following:

	1		1	1	
		1	0	0	1
+	1	1	0	1	1
1	0	0	1	0	0

Step 3: Calculate $1001_2 + 11011_2 = 100100_2$.

So, $1001_2 + 11011_2 = 100100_2$.

Subtraction in Bases Other Than Base 10

Subtraction in bases other than base 10 follow the same processes as base 10 subtraction, but, as with addition, using the addition table for the base.

Example 7 **Subtracting in Base 6**

Calculate $52_6 - 34_6$.

Solution

Step 1: Let's set up the subtraction using columns.

	5	2
−	3	4

Step 2: Just as we might do in base 10, we borrow a 1 from the 5 for the ones digit.

	$\cancel{5}4$	**12**
−	3	4

Step 3: Referring to the base 6 addition table , we see that $4 + 4 = 12$, so $12_6 - 4_6$ is 4_6.

	$\cancel{5}4$	**12**
−	3	4
		4

Step 4: Now we deal with the "tens" (really, sixes) digit, $4_6 - 3_6$, which equals 1_6 according to the base 6 addition table .

	$\cancel{5}4$	**12**
−	3	4
	1	4

So, $52_6 - 34_6 = 14_6$.

Example 8 **Subtracting in Base 12**

Calculate $A17_{12} - 4B3_{12}$.

Solution

Step 1: Let's set up the subtraction using columns.

	A	1	7
−	4	B	3

Step 2: Even in base 12, $7_{12} - 3_{12} = 4_{12}$.

	A	1	7
−	4	B	3
			4

Step 3: Moving to the "tens" digit, we have $1_{12} - B_{12}$. Since 1 is less than B in base 12, we need to borrow a 1 from the A, just as we would for subtraction in base 10.

	$\cancel{A}$**9**	**11**	7
−	4	B	3
			4

Step 4: According to the base 12 addition table in the solution for Example 4, $B_{12} + 2_{12} = 11_{12}$, so $11_{12} - B_{12} = 2_{12}$.

	$\cancel{A}$ 9	**1**1	7
−	4	B	3
		2	4

Step 5: Finally, we deal with the "hundreds" digit. According to the base 12 addition table in the solution for Example 4, $4_{12} + 5_{12} = 9_{12}$, so $9_{12} - 4_{12} = 5_{12}$.

	$\cancel{A}$ 9	11	7
−	4	B	3
	5	2	4

So, $A17_{12} - 4B3_{12} = 524_{12}$.

Errors When Adding and Subtracting in Bases Other Than Base 10

Errors when computing in bases other than 10 often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base. For instance, if a 9 shows up when working in base 7, you know an error has happened because 9 is not a legal symbol in base 7.

Example 9 **Identifying an Illegal Symbol in Arithmetic in a Base Other Than Base 10**

Explain the error in the following calculation:

$$15_6 + 34_6 = 49_6$$

Solution

Since the problem is in base 6, the symbol set available is 0, 1, 2, 3, 4 and 5. The 9 in the answer is clearly not a legal symbol for base 6. Looking back to the base 6 addition table , we see that $5_6 + 4_6 = 13_6$. Correcting the error, we see the sum is $15_6 + 34_6 = 53_6$.

The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, if you are working in base 13, then $9_{13} + 9_{13}$ is not 18_{13}, even though 18 is the correct answer in base 10.

Example 10 **Identifying an Arithmetic Error in a Base Other Than Base 10**

Explain the error in the following calculation, and correct the error:

$$89_{12} + 76_{12} = 165_{12}$$

Solution

If this problem was a base 10 problem, this would be the correct answer. However, in base 12, 9 + 6 is not 15, but is instead 13. To correct this error, carefully use the addition table for base 12. If properly used, the correct answer would be 143_{12}, as seen below:

		8	9
+		7	6
	1	4	3

Key Concepts

- Addition tables for bases other than 10 can be built using the same processes that are used in base 10, including using a number line.
- Addition in bases other than base 10 use the same processes as addition in base 10, but use the addition table for that base.
- Subtraction in bases other than base 10 use the same processes as subtraction in base 10, but use the addition table for that base.

4.5 Multiplication and Division in Base Systems

Figure 10: The processes for multiplication and division are the same for arithmetic in any bases.

Learning Objectives

After completing this section, you should be able to:

1. Multiply and divide in bases other than 10.
2. Identify errors in multiplying and dividing in bases other than 10.

Just as in Addition and Subtraction in Base Systems, once we decide on a system for counting, we need to establish rules for combining the numbers we're using. This includes the rules for multiplication and division. We are familiar with those operations in base 10. How do they change if we instead use a different base? A larger base? A smaller one?

In this section, we use **multiplication and division in bases** other than 10 by referencing the processes of base 10, but applied to a new base system.

Multiplication in Bases Other Than 10

Multiplication is a way of representing repeated additions, regardless of what base is being used. However, different bases have different addition rules. In order to create the multiplication tables for a base other than 10, we need to rely on addition and the addition table for the base. So let's look at multiplication in base 6.

Multiplication still has the same meaning as it does in base 10, in that 4×6 is 4 added to itself six times, $4 \times 6 = 4 + 4 + 4 + 4 + 4 + 4$.

So, let's apply that to base 6. It should be clear that 0 multiplied by anything, regardless of base, will give 0, and that 1 multiplied by anything, regardless of base, will be the value of "anything."

Step 1: So, we start with the table below:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4			
3	0	3				
4	0	4				
5	0	5				

Step 2: Notice $2 \times 2 = 4$ is there. But we didn't hit a problematic number there (4 works fine in both base 10 and base 6). But what is 2×3? If we use the repeated addition concept, $2 \times 3 = 2 + 2 + 2 = 4 + 2$. According to the base 6 addition table , $4 + 2 = 10$. So, we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10		
3	0	3	10			
4	0	4				
5	0	5				

Step 3: Next, we need to fill in 2×4. Using repeated addition, $2 \times 4 = 2 + 2 + 2 + 2 = 10 + 2 = 12$ (if we use our base 6 addition rules). So, we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	
3	0	3	10			
4	0	4	12			
5	0	5				

Step 4: Finally, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 12 + 2 = 14$. And so we add that to our table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10			
4	0	4	12			
5	0	5	14			

Step 5: A similar analysis will give us the remainder of the entries. Here is 4×5 demonstrated: $4 \times 5 = 4 + 4 + 4 + 4 + 4 = 12 + 12 + 4 = 24 + 4 = 32$.

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $4 + 4 = 12$, and then applying the addition processes we've always known, but with the base 6 table . In the end, our multiplication table is as follows:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32
5	0	5	14	23	32	41

Notice anything about that bottom line? Is that similar to what happens in base 10?

To summarize the creation of a multiplication in a base other than base 10, you need the addition table of the base with which you are working. Create the table, and calculate the entries of the multiplication table by performing repeated addition in that base. The table needs to be drawn only the one time.

Example 1 **Creating a Multiplication Table for a Base Lower Than 10**

Create the multiplication table for base 7.

Solution

Step 1: Let's apply the process demonstrated and outlined above to find the base 7 multiplication table. It should be clear that 0 multiplied by anything, regardless of base, will give 0, and that 1 multiplied by anything, regardless of base, will be the value of "anything." So, we start with the table below:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6			
3	0	3	6				
4	0	4					
5	0	5					
6	0	6					

Step 2: Notice $2 \times 2 = 4$ is there. But we didn't hit a problematic number there (4 works fine in both base 10 and base 6). The same is true for 2×3 and 3×2, which equal 6. But what is 2×4? If we use the repeated addition concept, $2 \times 4 = 2 + 2 + 2 + 2 = 6 + 2$. According to the base 7 addition table in the solution for Example 2 in Section 4.4, $6 + 2 = 11$. So, we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	**11**		
3	0	3	6				
4	0	4	**11**				
5	0	5					
6	0	6					

Step 3: Next, we need to fill in 2×5. Using repeated addition, $2 \times 5 = 2 + 2 + 2 + 2 + 2 = 11 + 2 = 13$ if we use our base 7 addition rules. So, we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	**13**	
3	0	3	6				
4	0	4	11				
5	0	5	**13**				
6	0	6					

Step 4: Finally, $2 \times 6 = 2 + 2 + 2 + 2 + 2 + 2 + 2 = 13 + 2 = 15$. And so we add that to our table:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	**15**
3	0	3	6				
4	0	4	11				
5	0	5	13				
6	0	6	**15**				

Step 5: A similar analysis will give us the remainder of the entries. Here is $4_7 \times 5_7$ demonstrated:

$$4_7 \times 5_7 = 4_7 + 4_7 + 4_7 + 4_7 + 4_7 = 11_7 + 11_7 + 4_7 = 22_7 + 4_7 = 26_7$$

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $4_7 + 4_7 = 11_7$ and then applying the addition processes we've always known, but with the base 7 table in the solution for Example 2 in Section 4.4. Using those addition rules, the rest of the table is given below:

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Example 2 **Creating a Multiplication Table for a Base Higher Than 10**

Create the multiplication table for base 12.

Solution

Let's apply the repeated addition to base 12. Here is $7_{12} \times 9_{12}$ demonstrated:

$$7_{12} \times 9_{12} = 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} + 7_{12} = 12_{12} + 12_{12} + 12_{12} + 12_{12} + 7_{12} = 48_{12} + 7_{12} = 53_{12}$$

This is done by using the addition rules from Addition and Subtraction in Base Systems, namely that $7_{12} + 7_{12} = 12_{12}$ and then applying the addition processes we've always known, but with the base 12 table in the solution for Example 4 in Section 4.4. Using those addition rules, the rest of the table is given below:

*	0	1	2	3	4	5	6	7	8	9	A	B
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	A	B
2	0	2	4	6	8	A	10	12	14	16	18	1A
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	A	13	18	21	26	2B	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2B	36	41	48	53	5A	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
A	0	A	18	26	34	42	50	5A	68	76	84	92
B	0	B	1A	29	38	47	56	65	74	83	92	A1

The multiplication table in base 2 below is as minimal as the addition table in the solution for . Since the product of 1 with anything is itself, the following multiplication table is found.

*	0	1
0	0	0
1	0	1

As with the addition table, we can use the multiplication tables and the addition tables to perform multiplication of two numbers in bases other than base 10. The process is the same, with the same carry rules and placeholder rules.

Example 3 **Multiplying in a Base Lower Than 10**

1. Calculate $45_6 \times 24_6$.
2. Calculate $101_2 \times 110_2$.

Solution

1. **Step 1:** Use the base 6 multiplication table and, when necessary, the base 6 addition table .

Set up this calculation using columns:

	4	5
x	2	4

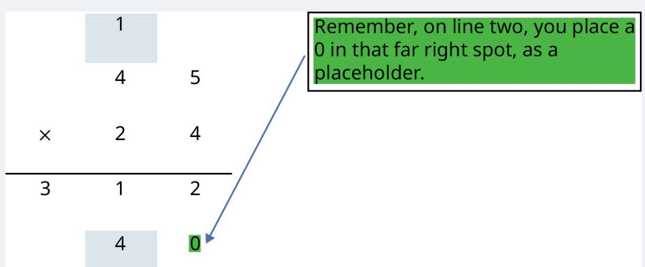
Step 2: Multiply the 1s digits, 5 and 4, using the base 6 multiplication table . There we see the result is 32_6. So, we enter the 2 and carry the 3.

	3	
	4	5
x	2	4
		2

Step 3: So, now we multiply the 4 and the 4, then add the 3 (just as you would do if multiplying two base 10 numbers!). $4_6 \times 4_6 = 24_6$ (from the base 6 table []), then $24_6 + 3_6 = 31_6$. So, we enter the 31.

	3	
	4	5
x	2	4
3	**1**	2

Step 4: Now we move on to the 2 in the "tens" place in the bottom value. We multiply the 2_6 and the 5_6, and we get 14_6. So, we enter the 4 and carry the 1.



Step 5: Next up, we multiply the 2 and the 4, and then add 1. This gives us $12_6 + 1_6 = 13_6$. We enter those on that second line.

		1	
		4	5
	x	2	4
	3	1	2
1	**3**	4	0

Step 6: Now we add down the columns.

		1	
		4	5
	x	2	4
1	3	1	2
1	3	4	0
2	0	5	2

Step 7: The 3 and the 3 add to 10 in base 6, so we enter the 0 and carry the 1. We now have the result:

$45_6 \times 24_6 = 2052_6$.

1. **Step 1:** Use the base 2 multiplication table and, when necessary, the base 2 addition table in the solution for Example 2 in Section 4.4. Set up this calculation using columns:

	1	0	1
x	1	1	0

Step 2: Using the pattern established above, and the processes from multiplication from base 10, we find the following:

			1	0	1
x			1	1	0
			0	0	0
		1	0	1	
	1	0	1		

Step 3: Adding down the columns results in the following:

			1	0	1
x			1	1	0
			0	0	0
		1	0	1	
	1	0	1		
	1	1	1	1	0

So, $101_2 \times 110_2 = 11110_2$.

Summarizing the process of multiplying two numbers in different bases, the multiplication table is referenced. Using that table, the multiplication is carried out in the same manner as it is in base 10. The addition rules for the base will also be referenced when carrying a 1 or when adding the results for each digit's multiplication line.

Example 4 **Multiplying in a Base Higher Than 10**

Calculate $3A_{12} \times 74_{12}$.

Solution

Step 1: Use the base 12 multiplication table in the solution for Example 2 and, when necessary, the base 12 addition table in the solution for . Set up this calculation using columns:

	3	A
×	7	4

Step 2: First, the 4 is multiplied by 3A, resulting in the first line.

	3	A
×	7	4
1	**3**	**4**

Step 3: Now we move on to the 7 in the "tens" place in the bottom value.

		5	
		3	A
	x	7	4
	1	3	4
2	**2**	**A**	**0**

Step 4: Now we add down the columns.

		3	A
	x	7	4
	1	3	4
2	2	A	0
2	**4**	**1**	**4**

Step 5: The 3 and the A add to 11 in base 12, so we enter the 1 and carry the 1.

We now have the result: $3A_{12} \times 74_{12} = 2414_{12}$.

Division in Bases Other Than 10

Just as with the other operations, **division in a base** other than 10, the process of division in a base other than 10 is the same as the process when working in base 10. For instance, $72 \div 9 = 8$ because, we know that $9 \times 8 = 72$. So, for many division problems, we are simply looking to the multiplication table to identify the appropriate multiplication rule.

Example 5 **Dividing with a Base Other Than 10**

1. Calculate $14_6 \div 5_6$.
2. Calculate $5A_{12} \div 7_{12}$

Solution

1. Looking at the multiplication table for base 6 , we see that $5_6 \times 2_6 = 14_6$. Using that, we know that $14_6 \div 5_6 = 2_6$.
2. Looking at the multiplication table for base 12 in the solution for Example 2, we see that $7_{12} \times A_{12} = 5A_{12}$. Using that, we know that $5A_{12} \div 7_{12} = A_{12}$.

Errors in Multiplying and Dividing in Bases Other Than Base 10

The types of errors encountered when multiplying and dividing in bases other than base 10 are the same as when adding and subtracting. They often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base.

Example 6 **Identifying an Illegal Symbol in a Base Other Than Base 10**

Explain the error in the following calculation, and determine the correct answer:

$$4_6 \times 2_6 = 8_6$$

Solution

Since the problem is in base 6, the symbol set available is 0, 1, 2, 3, 4, and 5. The 8 in the answer is clearly not a legal symbol for base 6. Looking back to the base 6 multiplication table , we see that $4_6 \times 2_6 = 12_6$.

The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, in base 17, $6_{17} \times 9_{17} = 54_{17}$ would be incorrect, even though in base 10, $6 \times 9 = 54$. That rule doesn't apply in base 17.

Example 7 **Identifying an Error in Arithmetic in a Base Other Than Base 10**

Explain the error in the following calculation. Determine the correct answer:

$$18_{12} \times 7_{12} = 126_{12}$$

Solution

If this problem was a base 10 problem, this would be the correct answer. However, in base 12, $8_{12} \times 7_{12}$ is not 56, but is instead 48. To correct this error, carefully use the multiplication table for base 12 . If properly used, the correct answer would be $18_{12} \times 7_{12} = \text{B}8_{12}$.

Key Concepts

- Multiplication tables for bases other than 10 can be built using the same processes that are used in base 10, including using repeated addition and the addition table for the base.
- Multiplication in bases other than base 10 use the same processes as multiplication in base 10, but use the multiplication table for that base.
- Basic division in bases other than base 10 use the same processes as basic division in base 10, where the missing factor process is used.

Projects

Additive Systems

Go online. Google “additive number systems.”What system comes up?

- Describe the additive system you found.

Using Google, identify three more additive systems of numbers.

- Compare and contrast the systems you found. For instance, how many times can a symbol be used before a new symbol is used.
- Identify three situations where additive systems are still used.

Computers and Bases

Use Google to determine what base computers use.

Were other bases attempted for use in computers?

Determine why the base used in computers is appropriate.

Determine how the base used in computers is related to the circuitry in computers.

Determine how Boolean logic and the base used in computers are related, and might be identical.

There is research into using quibits in computers. Find out what quibits are and how can they improve computing speed.

Cultures Using Base Systems Other Than 10

Using Google, find three cultures, other than Babylonian or Mayan, that use base systems other than 10.

- Tell what base is used for each system.
- If possible, determine why the culture used that base system.
- Choose one of those systems. Explain that base system. Be sure to address whether the system is additive, place-value based, a blend of the two, and if it employs a zero.

History of Zero

Using any resources available to you, determine the history of 0 in at least three different numbering systems. Address at least when and why such a development occurred and why a 0 is vital to the use of a positional system.

Numbering Systems from Other Global Regions

Using any resources available to you, find at least three numbering systems from sub-Saharan Africa, Australia, China, or the Pacific Islands. Explore if they are positional or additive systems (or combinations!), the terminology of the system, if they used a 0, and what base they employed (if positional).