Chapter 3: Real Number Systems and Number Theory

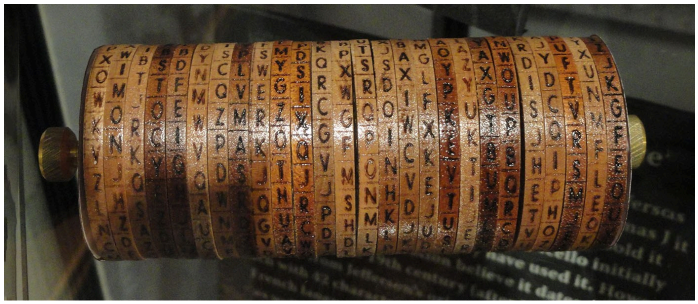


Figure 1: Encryption of computers and messages use very large prime numbers.

Encryption is used to secure online banking, for secure online shopping, and for browsing privately using VPNs (Virtual Private Networks). We need encryption (using prime numbers) for a secure exchange of information. For a prime number to be useful for encryption, though, it has to be large. Encryption uses a composite number that is the product of two very large primes. In order to break the encryption, one must determine the two primes that were used to form the composite number. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

3.1 Prime and Composite Numbers

Figure 2: Computers are protected using encryption based on prime numbers.

After completing this section, you should be able to:

1. Apply divisibility rules.
2. Define and identify numbers that are prime or composite.
3. Find the prime factorization of composite numbers.
4. Find the greatest common divisor.
5. Use the greatest common divisor to solve application problems.
6. Find the least common multiple.
7. Use the least common multiple to solve application problems.

   Encryption, which is needed for the secure exchange of information (i.e., online banking or shopping) is based on prime numbers. Encryption uses a composite number that is the product of two very large prime numbers. To break the encryption, the two primes that were used to form the composite number need to be determined. If the two prime numbers used are sufficiently large, even the fastest computer cannot determine those two prime numbers in a reasonable amount of time. It would take a computer 300 trillion years to crack the current encryption standard.

Applying Divisibility Rules

Before we begin our investigation of divisibility, we need to know some facts about important sets of numbers:

- The counting numbers are referred to as the **natural numbers**. This set of numbers, $\{1, 2, 3, 4, \ldots\}$, is denoted with the symbol $\mathbb{N}$.
- Another important set of numbers is the **integers**. The integers are the natural numbers, along with 0, and the negatives of the natural numbers. This set is often written as $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. We denote the integers with the symbol $\mathbb{Z}$.
- Notice that $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, or, $\mathbb{N} \subset \mathbb{Z}$. All the ideas of this section apply to the natural numbers, while only some apply to all the integers.

  Divisibility is when the integer n is divisible by m, if n can be written as m times another integer. Equivalently, there is no remainder when n is divided by m. There are many occasions when separating items into equal groups comes into play to ensure an equal distribution of whole items. For example, Francis, a preschool art teacher, has 15 students in one class. Francis has 225 sheets of construction paper and wants to provide each student with an equal number of pieces. To know if he will use all the construction paper, Francis is really asking if 225 can be evenly divided into 15 groups.

Example 1 **Determining if a Number Divides Another Number**

Determine if 36 is divisible by 4.

Solution

We could divide 36 by 4 and see if there is a remainder, or we could see if we can write

36 as 4 times another integer. If we divide 36 by 4, we see $36 \div 4 = 9$ with no remainder. We see that 36 is divisible by 4. We can write 36 as 4 times another integer, $36 = 4 \times 9$. By the definition of divisibility, 36 is divisible by 4.

You can quickly check if a number is divisible by 2, 3, 4, 5, 6, 9, 10, and 12. Each has an easy-to-identify feature, or rule, that indicates the divisibility by those numbers, as shown in the following table.

Divisor	Rule
2	Last digit is even
3	Add the digits of the number together. If that sum is divisible by 3, then so is the original number
4	Look at only the last two digits. If this number is divisible by 4, so is the original number
5	Look at only the last digit. If it is a 5 or a 0, then the original number is divisible by 5
6	If the number passes the rule for divisibility by 2 and for 3, then the number is divisible by 6
9	Add the digits of the number together. If that number is divisible by 9, then so is the original number
10	Look at only the last digit. If it is a 0, then the original number is divisible by 10
12	If the number passes the rule for 3 and 4, the number is divisible by 12

Example 2 **Using Divisibility Rules**

Using divisibility rules, determine if 245 is divisible by 5.

Solution

Since the last digit is a 5, the number 245 is divisible by 5 because the rule states if the last digit of the number is a 5 or a 0, then the original number is divisible by 5.

Example 3 **Using Divisibility Rules**

Using divisibility rules, determine if 25,983 is divisible by 9.

Solution

The divisibility rule for 9 is when the digits of the number are added, the sum is divisible by 9. So, we calculate the sum of the digits.

$2 + 5 + 9 + 8 + 3 = 27$. Since 27 is divisible by 9, so is the original number 25,983.

Example 4 **Using Divisibility Rules**

Can 298 coins be stacked into 6 stacks with an equal number of coins in each stack?

Solution

In order for the coins to be in equal-sized stacks, 298 would need to be divisible by 6. The divisibility rule for 6 is that the number passes the divisibility rules for both 2 and 3. Since the last digit is even, 298 is divisible by 2. To determine if 298 is divisible by 3, we first add the digits of the number: $2 + 9 + 8 = 19$. Since 19 is not divisible by 3, neither is 298. Because 298 is not divisible by 3, it is also not divisible by 6, which means they cannot be put into 6 equal stacks of coins.

Example 5 **Using Divisibility Rules**

Using divisibility rules, determine if 4,259 is divisible by 10.

Solution

The divisibility rule for 10 is that the last digit of the number is 0. Since the last digit of 4,259 is not 0, then 4,259 is not divisible by 10.

Example 6 **Using Divisibility Rules**

Using divisibility rules, determine if 936,276 is divisible by 4.

Solution

The divisibility rule for 4 is to check the last two digits of the number. If the number formed by the last two digits of the original number is divisible by 4, then so is the original number. The last two digits make the number 76 and 76 is divisible by 4, since $76 = 4 \times 19$. Since 76 is divisible by 4, so is 936,276.

VIDEO

Divisibility Rules

Prime and Composite Numbers

Sometimes, a natural number has only two unique divisors, 1 and itself. For instance, 7 and 19 are **prime**. In other words, there is no way to divide a prime number into groups with an equal number of things, unless there is only one group, or those groups have one item per group. Other natural numbers have more than two unique divisors, such as 4, or 26. These numbers are called **composite**. The number 1 is special; it is neither prime nor composite.

To determine if a number is prime or composite, you have to determine if the number has any divisors other than 1 and itself. The divisibility rules are useful here, and can quickly show you if a number has a divisor on that list.

However, if none of those divide the number, you still have to check all other possible prime divisors. What are the prime numbers that are possibly divisors of the number you are checking? You need only check the prime numbers up to the square root of the number in question. For instance, if you want to know if 2,117 is prime, you need to determine if any primes up to the square root of 2,117 (which is 46.0 when rounded to one decimal place) divide 2,117. If any of those primes are divisors of the number in question, then the number is composite. If none of those primes work, then the number is itself prime.

We can check divisibility with whatever tool we wish. Divisibility rules are quick for some prime divisors (2 and 5 come to mind) but aren't quick for other values (like 11). In place of divisibility rules, we could just use a calculator. If the prime number divides the number evenly (that is, there is no decimal or fractional part), then the number is divisible by that prime. is a quick list of the prime numbers up to 50. There are 15 prime numbers less than 50.

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47

Example 7 Determining If a Number Is Prime or Composite

Determine if 2,117 is prime or composite.

Solution

The square root of 2,117 is 46.0 (rounded to one decimal place). So, we need to check if 2,117 is divisible by any prime up to 46.

Step 1: First we'll use the rules of divisibility we learned earlier:

- We can tell 2,117 is not divisible by 2, as the last digit isn't even.
- 2,117 is not divisible by 5 (the last digit isn't 0 or 5).
- Add the digits of 2,117 to get 11, which is not divisible by 3. So, 2,117 is also not divisible by 3.

  Step 2: Now we repeat the process for all the primes up to 46.

  Using a calculator, we find that 2,117 divided by the prime numbers 7, 11, 13, 17, 19, and 23 results in a remainder, a decimal part. So, we know that 2,117 is not divisible by these prime numbers. (You should check these results yourself.)

  Moving on, we check the next prime: 29. Using the calculator to divide 2,117 by 29 results in 73. Since there is no decimal part, 2,117 is divisible by 29.

  This means that 2,117 is not a prime number, but rather, a composite number. Writing 2,117 as the product of 29 and another natural number, $2,117 = 29 \times 73$.

Example 8 Determining if a Number Is Prime or Composite

Determine if 423 is prime or composite.

Solution

The square root of 423 is 20.57 (rounded to two decimal places). So, we need to check if 423 is divisible by any prime up to 20.

Step 1: Check 2. We can tell 423 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is not divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. To check if 423 is divisible by 3, we use the divisibility rule for 3. When we take the sum of the digits of 423, the result is 9. Since 9 is divisible by 3, so is 423.

Since 423 is divisible by 3, then 423 is a composite number. Writing 423 as the product of 3 and another natural number, $423 = 3 \times 141$.

Example 9 **Determining if a Number Is Prime or Composite**

Determine if 1,034 is prime or composite

Solution

A quick inspection of 1,034 shows it is divisible by 2 since the last digit is even, and so 1,034 is a composite number.

Example 10 **Determining if a Number Is Prime or Composite**

Determine if 2,917 is prime or composite.

Solution

The square root of 2,917 is 50.01 (rounded to two decimal places). So, we need to check if 2,917 is divisible by any prime up to 50.

Step 1: Check 2. We can tell 2,917 is not divisible by 2, as the last digit isn't even.

Step 2: Check 5. It is it divisible by 5 (the last digit isn't 0 or 5).

Step 3: Check 3. Using the divisibility rule for 3, we take the sum of the digits of 2,917, which is 19. Since 19 is not divisible by 3, neither is 2,917.

Step 4: Check the rest of the primes up to 50 using a calculator. When 2,917 is divided by every prime number up to 50, the result has a decimal part.

Since no prime up to 50 divides 2,917, it is a prime number.

WHO KNEW?

ILLEGAL PRIMES

Large primes are a hot commodity. Using two very large primes (some have more than 22 million digits!) is necessary for secure encryption. Anyone who has a new prime that is large enough can use that prime to create a new encryption. Of course, whoever discovers a large prime could sell it to a security company. These primes are so useful for encryption, it is necessary to protect that intellectual property. In fact, at least one prime number was declared illegal.

VIDEO

Illegal Prime Number

PEOPLE IN MATHEMATICS

Sophie Germain

Figure 3: Sophie Germain

Born into a wealthy French family in 1776, Sophie Germain discovered and fell in love with mathematics by browsing her father's books. Clandestine study, hard and tenacious work, and a mathematical mindset did not lead to college, however, as she was not allowed to attend. She did manage, through friends, to obtain problem sets and submit brilliant solutions under the name Monsieur LaBlanc. One of her great interests was number theory, which is the study of properties of integers. One of her theorems, titled "Sophie Germain's Theorem," partially solved one of the great mathematical mysteries, Fermat's Last Theorem. From this she discovered what are now known as Sophie Germain Primes. A Sophie Germain Prime is a prime number that can be written in the form $2p + 1$, where p is a prime number. For instance, 23 is prime: $2(23) + 1 = 47$, which is prime, so 47 is a Sophie Germain Prime. It

> should be noted that $2p + 1$, where p is a prime number, may or may not be prime (check for $p = 7$!).

Finding the Prime Factorization of Composite Numbers

Before we can start with prime factorization, let's remind ourselves what it means to factor a number. We factor a number by identifying two (or more) numbers that, when multiplied, result in the original number. For instance, 3 and 24, when multiplied, give 72. So, 72 can be factored into 3×24. Notice that we could have factored the 72 differently, say as $72 = 6 \times 12$, or $72 = 2 \times 36$, or even as $72 = 3 \times 4 \times 6$.

Finding the **prime factorization** of a composite number means writing the number as the product of all of its prime factors. For example, $80 = 2 \times 2 \times 2 \times 2 \times 5$. Notice that all the numbers being multiplied on the right-hand side are prime numbers. Sometimes prime numbers repeat themselves in the factorization. When prime factors do repeat, we may write the prime factorization using **exponents**, as in $80 = 2^4 \times 5$. In that equation, the 2 is raised to the 4th power. The 4 is the exponent, and the 2 is the **base**. More generally, in the exponent notation a^b, the number represented by a is the base, and the number represented by b is the exponent.

One has to wonder if finding the prime factorization could result in different factorizations. The **Fundamental Theorem of Arithmetic** tells us that there is only one prime factorization for a given natural number.

Fundamental Theorem of Arithmetic

Every natural number, other than 1, can be expressed in exactly one way, apart from the arrangement, as a product of primes.

The process of finding the prime factorization of a number is iterative, which means we do a step, then repeat it until we cannot do the step any longer. The step we use is to identify one prime factor of the number, then write the number as the prime factor times another factor. We repeat this step on the other, newly found, factor. This step is repeated until no more primes can be factored from the remaining factor. This is easier to see and explain with an example.

Example 11 **Finding the Prime Factorization**

Find the prime factorization of 140.

Solution

Step 1: Identify a prime number that divides 140. Since 140 is even (the last digit is even), 2 divides 140. We then factor the 2 out of the 140, giving us $140 = 2 \times 70$.

Step 2: With the other factor, 70, find a prime factor of 70. Since 70 is also even, 2 divides 70. We factor the 2 out of the 70 and the factorization is now $140 = 2 \times 2 \times 35 = 2^2 \times 35$.

Notice that we expressed the two factors of 2 as 2^2.

Step 3: Look to the remaining factor, 35. The last digit of 35 is 5, so 5 is a factor of 35. We factor the 5 out of the 35. The factorization is now $140 = 2^2 \times 5 \times 7$.

Step 4: Look to the remaining factor, 7. Since 7 is prime, the process is complete.

The prime factorization of 140 is $2^2 \times 5 \times 7$.

Factor Trees

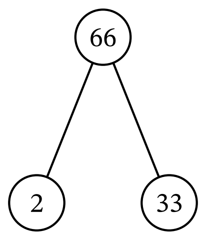
A useful tool for helping with prime factorization is a **factor tree**. To create a factor tree for the natural number n (where n is not 1), perform the following steps:

Step 1: If n is prime, you're done. If n is composite, continue to the next step.

Step 2: Identify two divisors of n, call them a and b.

Step 3: Write the number n down, and draw two branches extending down (or to the right) of the number n.

Step 4: Label the end of one branch a, the other as b.



Step 5: If a and b are prime, the tree is complete. When a number at this step is a prime number, we refer to it as a leaf of the tree diagram.

Step 6: If either a or b are composite, repeat Steps 2 through 4 for a and b.

Step 7: The process stops when the leaves are all prime.

Step 8: The prime factorization is then the product of all the leaves.

This is best seen in an example.

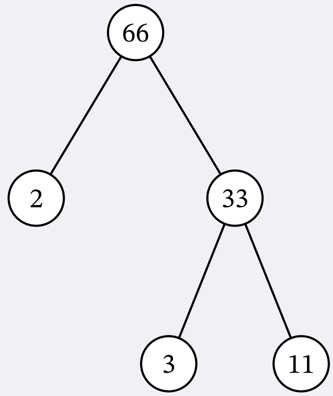
Example 12 **Finding the Prime Factorization**

Find the prime factorization of 66.

Solution

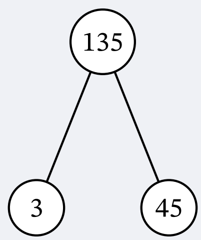
Since 66 is even, 2 is a factor.

Step 1: Factor out the 2. The factorization is $66 = 2 \times 33$. The factor tree is shown.



Since 2 is a factor, that branch is done, and 2 is a leaf.

Step 2: The 33, though, is divisible by 3, and is the product of 3 and 11. We attach that to the factor tree .



Since the 2, 3 and 11 are all prime, the factor tree is done.

The prime factorization of 66 is the product of the leaves, so $66 = 2 \times 3 \times 11$. The factorization is complete.

VIDEO

Using a Factor Tree to Find the Prime Factorization

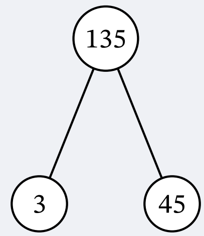
Example 13 **Finding the Prime Factorization**

Find the prime factorization of 135.

Solution

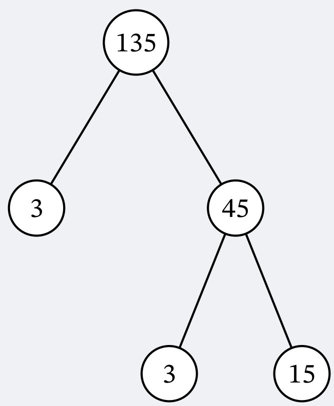
The number 135 is divisible by 3, and so 3 is a factor of 135.

Step 1: Factor out the 3. The factorization is $135 = 3 \times 45$. Using the factor tree ,



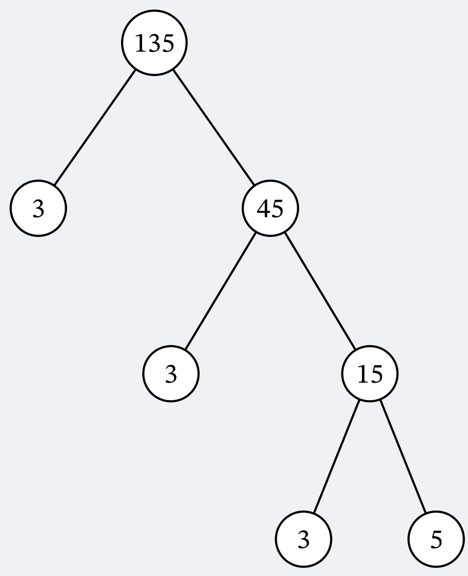
45 is also divisible by 3.

Step 2: Factor out a 3 from 45. The other factor is 15. The factor tree is shown.



15 is also divisible by 3.

Step 3: The factors of 15 are 3 and 5. The factor tree is shown.



All the leaves are prime, so the process is complete. The prime factorization of 135 is $3^3 \times 5$.

Example 14 **Identifying Prime Factors**

How many different prime factors does 10,241 have?

Solution

To know how many different prime factors 10,241 has, we need the prime factorization of 10,241.

Step 1: Use divisibility rules, to see that the number 10,241 is not divisible by 2 or by 3 (the sum of the digits is 8), or by 5. However, it is divisible by 7.

Step 2: After factoring the 7, the factorization is $10,241 = 7 \times 1463$ 1,463 is also divisible by 7.

Step 3: After factoring the 7, the factorization is $10,241 = 7 \times 7 \times 209 = 7^2 \times 209$. The number 209 is not divisible by 7.

Step 4: Check the next prime number: 11; 11 does divide 1,463.

Step 5: After factoring the 11, the factorization is $10,241 = 7^2 \times 11 \times 19$.

Since 19 is prime, the prime factorization of 10,241 is complete. We see that 10,241 has three different prime factors: 7, 11, and 19.

VIDEO

Finding the Prime Factorization of 168

TECH CHECK

Using Wolfram Alpha to Find Prime Factorizations

The Wolfram Alpha website is a powerful resource available for free to use. It is designed using AI so that it understands natural language requests. For instance, typing the question "What is the prime factorization of 543,390?" gets a rather quick answer of $2 \times 3 \times 5 \times 59 \times 307$. So, if you want to find the prime factorization of a number, you can simply ask Wolfram Alpha to find the prime factorization of the number.

Finding the Greatest Common Divisor

Two numbers often have more than one divisor in common (all pairs of natural numbers have the common divisor 1). When listing the common divisors, it's often the case that the largest is of interest. This divisor is called the **greatest common divisor** and is denoted **GCD**. It is also sometimes referred to as the greatest common factor (GCF).

For instance, 6 is the greatest common divisor of 12 and 18. We can see this by listing all the divisors of each number and, by inspection, select the largest value that shows up in each list.

The divisors of 12	1, 2, 3, 4, 6, 12
The divisors of 18	1, 2, 3, 6, 9

It is easy to see that 6 is the largest value that appears in both lists.

Example 15 **Finding the Greatest Common Divisor Using Lists**

Find the greatest common divisor of 1,400 and 250 by listing all divisors of each number.

Solution

We create a list of all the divisors of 1,400 and of 250, and choose the largest one.

The divisors of 1,400 are

1, 2, 4, 5, 7, 8, 10, 14, 20, 25, 28, 35, 40, 50, 56, 70, 100, 140, 175, 200, 280, 350, 700, 1,400.

The divisors of 250 are

1, 2, 5, 10, 25, 50, 125, 250.

The largest value that appears on both lists is 50, so the greatest common divisor of 1,400 and 250 is 50.

Listing all the divisors of the numbers in the set will always work, but for some relatively small numbers, the set of all divisors can become pretty big, and finding them all can be a chore. Another approach to finding the greatest common divisor is to use the prime factorization of the numbers. To do so, use the following steps:

Step 1: Find the prime factorization of the numbers.

Step 2: Identify the prime factors that appear in every number's prime factorization. These are called the common prime factors.

Step 3: Identify the smallest exponent of each prime factor identified in Step 2 in the prime factorizations.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3. The result is the greatest common divisor.

Example 16 **Finding the Greatest Common Divisor Using Prime Factorization**

Find the greatest common divisor of 1,400 and 250 by using their prime factorizations.

Solution

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 1,400 is $2^3 \times 5^2 \times 7$.

The prime factorization of 250 is $250 = 2 \times 5^3$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2 and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponent of common prime factor 2 in the prime factorization of 1,400 is 3, and in the prime factorization of 250 is 1. The smallest of those exponents is 1.

The exponent of the common prime factor 5 in the prime factorization of 1,400 is 2 and is in the prime factorization of 250 is 3. The smallest of these exponents is 2.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 5^2 = 50$. The greatest common divisor of 1,400 and 250 is $2 \times 5^2 = 50$.

Notice that the answer matches the one we found in Example 3.15.

Example 17 **Finding the Greatest Common Divisor Using Prime Factorization**

Find the greatest common divisor of 600 and 784 by using their prime factorizations.

Solution

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 600 is $2^3 \times 3 \times 5^2$.

The prime factorization of 784 is $2^4 \times 7^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

There is only one common prime factor, 2.

Step 3: Identify the smallest exponent of each prime from identified in Step 2 in the prime factorizations.

The exponent of 2 in the prime factorization of 600 is 3. The exponent of 2 in the prime factorization of 784 is 4. So, the smallest exponent of 2 is 3.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^3 = 8$. The greatest common divisor of 600 and 784 is 8.

PEOPLE IN MATHEMATICS

Srinivasa Ramanujan

Figure 10: Srinivasa Ramanujan

Ramanujan was born in southern India in 1887, during British rule. He was a self-taught mathematician, who, while in high school, began working through a two-volume text of mathematical theorems and results. His work included examination of the distribution of primes. He eventually came to the attention of British mathematician, G.H. Hardy. During one visit, Hardy mentioned to Ramanujan that his taxicab number was 1,729, remarking that 1,729 appeared to be a rather dull number. To which Ramanujan responded, "It is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."

TECH CHECK

Using Desmos to Find the GCD

To find the GCD of a set of numbers in Desmos, type gcd(first_number,second_number…) and Desmos will display the GCD of the numbers, as the numbers are typed!

VIDEO

Using Desmos to Find the GCD

Applications of the Greatest Common Divisor

The greatest common divisor has uses that are related to other mathematics (reducing fractions) but also in everyday applications. We'll look at two such applications, which have very similar underlying structures. In each case, something must divide the groups or measurements equally.

Example 18 **Calculating Floor Tile Size**

Suppose you have a rectangular room that is 570 cm wide and 450 cm long. You wish to cover the floor of the room with square tiles. What's the largest size square tile that can be used to cover this floor?

Solution

Using squares means that the length and width of the tiles are equal. To ensure we are using full tiles, the side length of the square tiles must divide the length of the room and the width of the room. Since we want the largest square tiles, we need the GCD of the width and length of the room or the GCD of 570 and 450.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 570 is $2 \times 3 \times 5 \times 19$.

The prime factorization of 450 is $2 \times 3^2 \times 5^2$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2, 3, and 5.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The exponents of 2 in the prime factorizations of 570 and 450 are 1 and 1. So the smallest exponent for 2 is 1.

The exponents of 3 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 3 is 1.

The exponents of 5 in the prime factorizations of 570 and 450 are 1 and 2, so the smallest exponent for 5 is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 \times 5^1 = 30$. The GCD of 450 and 570 is 30, so the largest size square tile that can be used to cover the floor is 30 cm by 30 cm.

Example 19 **Organizing Books Per Shelf**

Suppose you want to organize books onto shelves, and you want the shelves to hold the same number of books. Each shelf will only contain one genre of book. You have 24 sci-fi, 42 fantasy, and 30 horror books. How many books can go on each shelf?

Solution

Since we want shelves that hold an equal number of books, and a shelf can only hold one genre of book, we need a number that will equally divide 24, 42, and 30. So, we need the GCD of the number of books of each genre or the GCD of 24, 42, and 30.

Step 1: Find the prime factorizations of the numbers.

The prime factorization of 24 is $2^3 \times 3$.

The prime factorization of 42 is $2 \times 3 \times 7$.

The prime factorization of 30 is $2 \times 3 \times 5$.

Step 2: Identify the prime factors that appear in every number's prime factorization.

The common prime factors are 2 and 3.

Step 3: Identify the smallest exponent of each prime identified in Step 2 in the prime factorizations.

The smallest exponent of 2 and 3 in the factorizations is 1.

Step 4: Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

This gives $2^1 \times 3^1 = 6$. The GCD of 24, 42, and 30 is 6, so the largest number of books that can go on a shelf is 6.

VIDEO

Applying the GCD

Finding the Least Common Multiple

The flip side to a divisor of a number is a **multiple** of a number. For example, 5 is a divisor of 45 and so 45 is a multiple of 5. More generally, if the number a divides the number b, then b is a multiple of a.

This drives the idea of the least common multiple of a set of numbers. A common multiple of a set of numbers is a multiple of each of those numbers. For instance, 45 is a common multiple of 9 and 5, because 45 is a multiple of 9 (9 divides 45) and 45 is also a multiple of 5 (5 divides 45). The **least common multiple** (LCM) of a set of number is the smallest positive common multiple of that set of numbers.

There are (at least) three ways to find the LCM of a set of numbers, and we will explore two of them. One way is to create a list of multiples of each number in the set, and then identify the smallest multiple that appears in those lists.

Example 20 **Finding the Least Common Multiple Using Lists**

Find the LCM of 24 and 90 by listing multiples and choosing the smallest common multiple.

Solution

Create a list of the multiples of each number.

Step 1: The first 15 multiples for 24:

24, 48, 72, 96, 120, 144, 168, 192, 216, 240, 264, 288, 312, 336, 360.

Step 2: The first 15 multiples for 90:

90, 180, 270, 360, 450, 540, 630, 720, 810, 900, 990, 1,080, 1,170, 1,260, 1,350.

There is one multiple common to these lists, which is 360. So, 360 is the LCM of 24 and 90.

The second method we can use is to find the prime factorizations of the number in the set to build the LCM of the numbers based on the prime divisors of the numbers. The LCM can be built from the prime factorization of the numbers in the set in a similar way as when finding the greatest common divisor. Here are the steps for using the prime factorization process for finding the LCM.

Step 1: Find the prime factorization of each number.

Step 2: Identify each prime that is present in any of the prime factorizations.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Step 4:. Multiply the prime factors identified in Step 2 raised to the powers identified in Step 3.

Example 21 **Finding the Least Common Multiple Using Prime Factorization**

Use the prime factorizations of 24 and 90 to identify their LCM.

Solution

Step 1: Find the prime factorization of each number.

$$24 = 2^3 \times 3$$

$$90 = 2 \times 3^2 \times 5$$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, and 5.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5
Largest exponent	3	2	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 24 and 90 is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5^1 = 360$.

Example 22 **Finding the Least Common Multiple Using Prime Factorization**

Use the prime factorizations of 36, 66, and 250 to identify the LCM.

Solution

Step 1: Find the prime factorization of each number.

$$36 = 2^2 \times 3^2$$

$$66 = 2 \times 3 \times 11$$

$$250 = 2 \times 5^3$$

Step 2: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5, and 11.

Step 3: Identify the largest exponent of each prime identified in Step 2 in the prime factorizations.

Prime	2	3	5	11
Largest exponent	2	2	3	1

Step 4: Compute the LCM by multiplying the prime factors identified in Step 2 raised to the powers identified in Step 3.

The LCM for 36, 66, and 250 is $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 11 = 2^2 \times 3^2 \times 5^3 \times 11^1 = 49,500$.

Using lists for three or more numbers, particularly larger numbers, could take quite a bit of time. Frequently, as in this example, the prime factorization process is much quicker. In practice, you can use either listing or prime factorization to find the LCM.

Example 23 **Finding the Least Common Multiple Using Both Methods**

Find the LCM of 20, 36, and 45 using lists and prime factorization.

Solution

Step 1: Use listing. List the multiples:

20	20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220
36	36, 72, 108, 144, 180, 216, 252, 288, 324, 360
45	45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495, 540

The smallest value that appears on all the lists is 180, so 180 is the LCM of 20, 36, and 45.

Step 2: Find the prime factorization of each number.

$$20 = 2^2 \times 5$$

$$36 = 2^2 \times 3^2$$

$$45 = 3^2 \times 5$$

Step 3: Identify each prime that is present in any of the prime factorizations.

The prime numbers present in the prime factorizations are 2, 3, 5.

Step 4: Identify the largest exponent of each prime identified in Step 3 in the prime factorizations.

Prime	2	3	5
Largest exponent	2	2	1

Step 5: Compute the LCM by multiplying the prime factors identified in Step 3 raised to the powers identified in Step 4.

The LCM for 20, 36, and 45 is $2^2 \times 3^2 \times 5^1 = 180$.

Both listing and prime factorization produced the same result: the LCM is 180.

VIDEO

Finding the LCM

TECH CHECK

Using Desmos to Find the LCM

To find the LCM of a set of numbers in Desmos, type lcm(first_number,second_number...) and Desmos will display the LCM of the numbers, as the numbers are typed!

VIDEO

Using Desmos to find the LCM

Applications of the Least Common Multiple

Some applications of LCM involve events that repeat at fixed intervals, such as visits to a location. Other applications involve getting things to be of equal magnitude when using parts that are different sizes (see Example 25, for instance). In each case, we may be looking to determine when two processes "line up."

Example 24 **Determining Scheduling Overlap Using the Least Common Multiple**

Two students, João, and Amelia, meet one day at an assisted living facility where they volunteer. João volunteers every 6 days. Amelia volunteers every 10 days. How many days will it be until they are both volunteering on the same day again?

Solution

If we list the days that each student will volunteer, it becomes clear that we could solve this problem using the LCM of 6 and 10.

João will be at the assisted living facility 6, 12, 18, 24, 30, 36, 42, and 48 days later.

Amelia will be at the assisted living facility 10, 20, 30, 40, 50, and 60 days later.

The smallest number appearing on both lists is 30. João and Amelia will once more be volunteering together 30 days later.

Example 25 **Determining the Minimum Height Using the LCM**

A team-building exercise has teams build a house of cards as high as possible. However, the cards for different teams are of different sizes. Team 1 uses 10 cm × 10 cm cards, while Team 2 uses 8 cm × 8 cm cards. What is the minimum height when the two teams will be tied? Ignore the width of the cards.

Solution

This is an example where we want to put together objects with different sizes. We want to know the minimum height when they are tied, or when the houses of cards line up the first time. The heights of the towers built using the 10 cm × 10 cm cards will be 10, 20, 30, 40, 50, and 60 cm tall. When the 8 cm × 8 cm cards are used, the tower will be 8, 16, 24, 32, 40, 48, and 56 cm tall. The smallest number appearing on both lists is 40. The first time they are tied is when the two towers are 40 cm tall.

VIDEO

Application of LCM

WORK IT OUT

Prime Number Life Cycles—Cicadas

Cicadas are known to have life cycles of 13 or 17 years, which are prime numbers. Why would a prime number life cycle be an evolutionary advantage? To figure this out, we have to explore how common multiples work with prime numbers.

Make a conjecture regarding the LCM of a prime number and another number. Test this conjecture with a few examples of your own making.

Key Terms

- natural numbers
- factor of a number
- multiple of a number
- prime number
- composite number
- prime factorization
- greatest common divisor (GCD)
- least common multiple (LCM)

Key Concepts

- The natural numbers can be categorized as 1, prime numbers, and composite numbers.
- Prime numbers have as their only factors 1 and themselves.
- Composite numbers have at least three distinct factors.
- Composite numbers can be written in their prime factorization form, which is found by repeatedly factoring prime factors from the number.
- The greatest common divisor (GCD) of a set of numbers is the largest integer that divides all of the numbers in the set. The prime factorizations of the numbers can be used to identify the greatest common divisor.
- The least common multiple (LCM) of a set of numbers is the smallest integer that is divisible by all of the numbers in the set. The prime factorizations of the numbers can be used to identify the least common multiple.
- There are various ways that the GCD and LCM are applied.

Videos

- Divisibility Rules
- Illegal Prime Number
- Using a Factor Tree to Find the Prime Factorization
- Finding the Prime Factorization of 168
- Using Desmos to find the GCD
- Applying the GCD
- Finding the LCM
- Using Desmos to find the LCM
- Application of LCM

3.2 The Integers

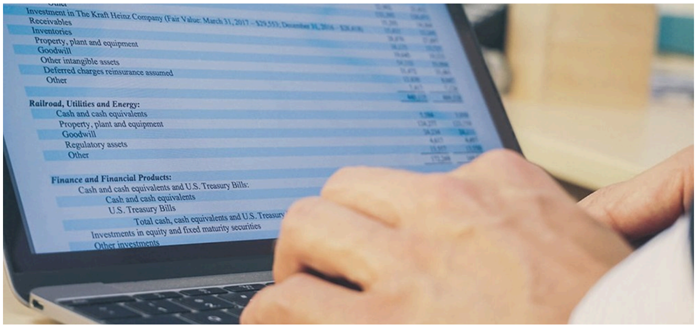


Figure 11: A ledger comparing assets to debts, resulting in net wealth.

After completing this section, you should be able to:

1. Define and identify numbers that are integers.
2. Graph integers on a number line.
3. Compare integers.
4. Compute the absolute value of an integer.
5. Add and subtract integers.
6. Multiply and divide integers.

   Positive net wealth is when the total value of a person's assets, such as their home, their 401(k), their car, and savings account balance, exceed that of their debts, such as car loans, mortgages, or credit card debt. However, when the total value of debt exceeds the total value of assets, then the person has negative net wealth. Expressing the negative net wealth as a negative number allows people to work with the positive net values and negative net values with the same mathematical processes, and in the same applications. This section introduces the integers and operations with integers.

Defining and Identifying Integers

Extending the counting numbers to include negative numbers and zero forms the **integers**. Any other number that cannot be written as $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is not an integer.

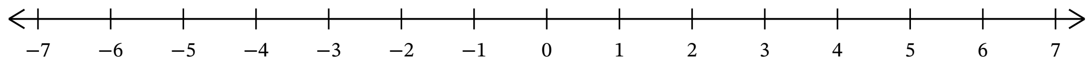
Example 1 **Identifying Integers**

Which of the following are integers and which are not?

-3	Is an integer, as it is the negative of a counting number
$\sqrt{24}$	This is not written as an integer. Entering the square root of 24 in a calculator, such as desmos, the result is 4.899 (rounded off). Since this is not an integer, then $\sqrt{24}$ is not an integer.
36/4	Since 36 divided by 4 is 9, and 9 is an integer, then 36/4 is an integer.
45	Is an integer, as it is a counting number
63.9	Is not an integer, because it is not a counting number and not the negative of a counting number.
2/7	Dividing 2 by 7 results in a number less than 1, but greater than 0, so is between two consecutive integers. So, 2/7 is not an integer.
-16.0	Is an integer, since the decimal part is 0

Graphing Integers on a Number Line

Integers are often imagined as steps along a path. You start at 0, and going to the left is going backward, or in the negative direction, while going to the right is going forward, or in the positive direction. A number line helps envision the integers. This also means that an integer gives magnitude (size) and direction (positive is to the right, negative is to the left). Graphing an integer on the number line means placing a solid dot at the integer on the number line.



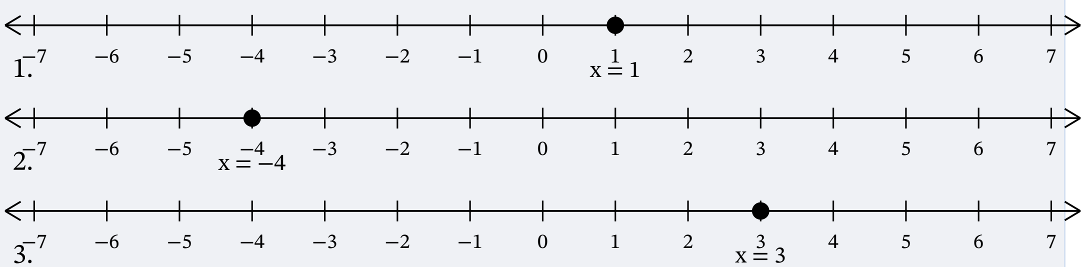
VIDEO
Graphing Integers on the Number Line

Example 2 **Graphing Numbers on the Number Line**

Graph the following on the number line:

1. 1
2. −4
3. 3

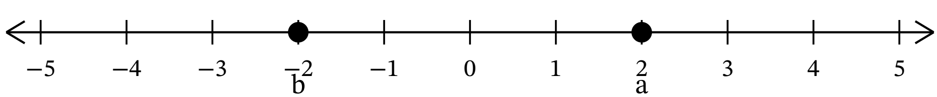
Solution



Comparing Integers

When determining if one quantity or size is larger than another, we know it means there is more of whatever is being discussed. In terms of positive integers, we can envision that larger integers are further to the right on the number line. This idea applies to negative integers also. This means that ***a* is greater than *b*** when a is to the right of b on the number line. We write $a > b$. When a is greater than b, we can also say that ***b* is less than *a***. On the number line, b would be to the left of a. We write $b < a$.

We need to recognize that $a > b$ means the same thing as $b < a$. This can be seen on the number line in . On this number line, a is to the right of b, so $a > b$. But this means b is to the left of a, so $b < a$.



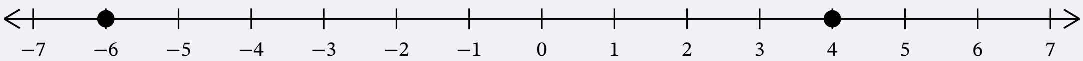
VIDEO
Comparing Integers Using the Number Line

Example 3 **Comparing Integers Using a Number Line**

Determine which of -6 and 4 is larger using a number line, and express that using both the greater than and the less than notations.

Solution

To illustrate this, we use a number line .



Since -6 is to the left of 4, then -6 is less than 4. We can write this as $-6 < 4$. Another way of expressing this is that 4 is greater than -6. So we can also write $4 > -6$.

Example 4 **Comparing Negative Integers**

Determine which of -6 and -2 is larger, and express that using both the greater than and the less than notations.

Solution

To illustrate this, we use a number line .

Since -6 is to the left of -2, then -6 is less than -2. We can write this as $-6 < -2$.

Another way of expressing this is that -2 is greater than -6. So we can also write $-2 > -6$.

CHECKPOINT

Warning: People often ignore the negative signs, and think than since 6 is greater than 2, -6 is greater than -2. To avoid that error, remember that the greater number is to the right on the number line.

Example 5 **Comparing Integers by Quantity**

Determine which of 27 and 410 is larger, and express that using both the greater than and the less than notations.

Solution

When thinking about quantity, 410 is more than 27. So, 410 is greater than 27 and 27 is less than 410. We can write this as $410 > 27$ or as $27 < 410$.

The Absolute Value of an Integer

When talking about graphing integers on the number line, one interpretation suggests it is like walking along a path. Negative is going to the left of 0, and positive going to the right. If you take 30 steps to the right, you are 30 steps away from 0. On the other hand, when you take 30 steps to the left, you are still 30 steps away from 0. So, in a way, even though one is negative and the other positive, these two numbers, 30 and -30, are equal since both are 30 steps away from

0. The **absolute value of an integer** n is the distance from n to 0, regardless of the direction. The notation for absolute value of the integer n is $| n |$.

If we think of an integer as both direction and magnitude (size), absolute value is the magnitude part.

Calculating the absolute value of an integer is very straightforward. If the integer is positive, then the absolute value of the integer is just the integer itself. If the integer is negative, then to compute the absolute value of the integer, simply remove the negative sign. Keeping in mind the number line as a path, when you've gone 10 steps to the left of 0, you have still taken 10 steps, and the direction does not matter.

VIDEO

Evaluating the Absolute Value of an Integer

Example 6 **Calculating the Absolute Value of a Positive Integer**

Calculate $|19|$.

Solution

Since the number inside the absolute value symbol is positive, the absolute value is just the number itself. So $|19| = 19$.

Example 7 **Calculating the Absolute Value of a Negative Integer**

Calculate $|-435|$.

Solution

Since the number inside the absolute value is negative, the absolute value removes the negative sign. So $|-435| = 435$.

Adding and Subtracting Integers

You may recall having approached adding and subtracting integers using the number line from earlier in your academic life. Adding a positive integer results in moving to the right on the number line. Adding a negative integer results in moving to the left. Subtracting a positive integer results in a move to the left on the number line. But subtracting a negative integer results in a move to the right.

This leads to a few adding and subtracting rules, such as:

Rule 1: Subtracting a negative is the same as adding a positive.

Rule 2: Adding two negative integers always results in a negative integer.

Rule 3: Adding two positive integers always results in a positive integer.

Rule 4: The sign when adding integers with opposite signs is the same as the integer with the larger absolute value.

These rules are good to keep in the back of your mind, as they can serve as a quick error check when you use a calculator.

Example 8 **Adding Integers**

Use your calculator to calculate $4 + (-7)$. Explain how the answer agrees with what was expected.

Solution

Using a calculator, we find that $4 + (-7) = -3$. Since we are adding integers with opposite signs, the sign of the answer matches the sign of the integer with the larger absolute value which $|-7|=7$.

Example 9 **Subtracting Positive Integers**

Use your calculator to calculate $18 - 9$. Explain how the answer agrees with what was expected.

Solution

Using a calculator, we find that $18 - 9 = 9$. Since 18 was larger than 9, we expected the difference to be positive.

Example 10 **Subtracting with Negative Integers**

Use your calculator to calculate $27 - (-13)$. Explain how the answer agrees with what was expected.

Solution

Using a calculator, we find that $27 - (-13) = 40$. Since we're subtracting a negative number, it is the same as adding a positive, so this is the same as $27 + 13 = 40$.

Example 11 **Adding Integers with Opposite Signs**

Use your calculator to calculate $(-13) + 90$. Explain how the answer agrees with what was expected.

Solution

Using a calculator, we find that $(-13) + 90 = 77$. Since we are adding integers with opposite signs, the sign of the answer matches the sign of the integer with the larger absolute value, which is positive since 90 is positive.

One use of negative numbers is determining **net worth**, which is all the weath someone owns less all that someone owes. Sometimes net worth is positive (which is good), and sometimes net worth is negative (which can be stressful).

Example 12 **Calculating Net Worth**

Jennifer is owed \$50 from her friend Janice, but owes her friend Pat \$87. What is Jennifer's net worth?

Solution

Net worth is the amount that one is owed minus the amount one owes. Jennifer is owed \$50 but owes \$87. So, her net worth is \$50 – \$87 = –\$37. The negative indicates that Jennifer owes more than she is owed.

Multiplying and Dividing Integers

Similar to addition and subtraction, the signs of the integers impact the results when multiplying and dividing integers. The rules are fairly straightforward, but again rely on the direction on the number line. There are only two rules.

Rule 1: When multiplying or dividing two integers with the same sign, the result is positive.

Rule 2: When multiplying or dividing two integers with opposite signs, the result is negative.

Just as before, these rules can serve as a quick error check when using a calculator.

Example 13 **Multiplying Positive Integers**

Use your calculator to calculate 4×8. Explain how the answer agrees with what was expected.

Solution

Entering 4×8 into your calculator, the result is 32. This agrees with our expectation. The numbers have the same signs, so the result is positive.

Example 14 **Multiplying Integers with Different Signs**

Use your calculator to calculate $9 \times (-10)$. Explain how the answer agrees with what was expected.

Solution

Entering $9 \times (-10)$ into your calculator, the result is -90. This agrees with our expectation. The numbers have opposite signs, so the result is negative.

Example 15 **Dividing Integers with Different Signs**

Use your calculator to calculate $400/(-25)$. Explain how the answer agrees with what was expected.

Solution

Entering $400/(-25)$ into your calculator, the result is -16. This agrees with our expectation. The numbers have opposite signs, so the result is negative.

Example 16 **Dividing Negative Integers**

Use your calculator to calculate $-750/(-3)$. Explain how the answer agrees with what was expected.

Solution

Entering $-750/(-3)$ into your calculator, the result is 250. This agrees with our expectation. The numbers have the same signs, so the result is positive.

At the end of a season, a team may wish to buy their coach an end-of season gift. It makes sense to share the cost equally among the members. To do so, the team would need to find the average (or mean) cost per member. The **average (or mean) of a set of** numbers is the sum of the numbers divided by the number values that are being averaged.

Example 17 **Finding the Average of a Set of Numbers**

The daily low temperatures in Barrie, Ontario, for the week of February 14, 2021, were $-20°$, $-12°$, $-15°$, $-23°$, $-17°$, $-13°$, and $-19°$ degrees Celsius. What was the average daily temperature for the week of February 14, 2021, in Barrie?

Solution

Step 1: To find the average daily temperature, we first need to add the temperatures.

$(-20) + (-12) + (-15) + (-23) + (-17) + (-13) + (-19) = -119$

Step 2: That sum will then be divided by 7 since we are averaging over seven days, giving $-119/7 = -17$. So, the average daily temperature in Barrie, Ontario the week of February 14, 2021, was $-17°$ Celsius.

Key Terms

- integer
- absolute value
- average of a set of numbers

Key Concepts

- A set of numbers that can be built from the natural numbers are the integers, which consist of the natural numbers, zero (0), and the negatives of the natural numbers.
- Integers are often graphed on a number line, which helps display the relative positions and values of those numbers.
- The number line can be used to visualize when one integer is larger than or smaller than another integer.
- Arithmetic operations with integers are similar to the operations with natural numbers, except that the sign (positive or negative) of the numbers will determine the sign (positive or negative) of the result.

Videos

- Graphing Integers on the Number Line
- Comparing Integers Using the Number Line
- Evaluating the Absolute Value of an Integer

3.3 Order of Operations

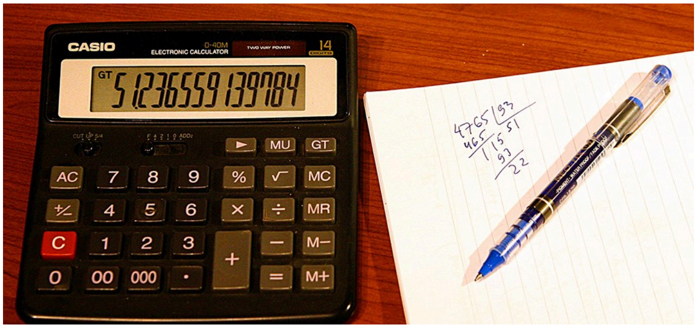


Figure 19: Calculators may automatically apply order of operations to calculations.

After completing this module, you should be able to:

1. Simplify expressions using order of operations.
2. Simplify expressions using order of operations involving grouping symbols.

Calculates else sure someone be rules expect explicit we what that needs to need make, we to that them what be calculate to calculated first.

You probably read that sentence and couldn't make heads or tails of it. Seems like it might concern calculations, but maybe it concerns needs? You may even be attempting to unscramble the sentence as you read it, placing words in the order you might expect them to appear in. The reason that the sentence makes no sense is that the words don't follow the order you expect them to follow. Unscrambled, the sentence was intended to be "To be sure that someone else calculates when we expect them to calculate, we need rules that make explicit what needs to be calculated first."

Similarly, when working with math expressions and equations, if we don't follow the rules for **order of operations**, arithmetic expressions make no sense. Just a simple expression would be problematic if we didn't have some rules to tell us what to calculate first. For instance, $4 \times 2^2 + 3 + 5^2$ can be calculated in many ways. You could get 5,184. Or, you could get 80. Or, 96. The issue is that without following a set of rules for calculation, the same expression will give various results. In case you are curious, using the appropriate order of operations, we find $4 \times 2^2 + 3 + 5^2 = 44$.

Simplify Expressions Using Order of Operations

The order in which mathematical operations is performed is a convention that makes it easier for anyone to correctly calculate. They follow the acronym EMDAS:

E	Exponents
M/D	Multiplication and division
A/S	Addition and Subtraction

So, what does EMDAS tell us to do? In an equation, moving left to right, we begin by calculating all the exponents first. Once the exponents have been calculated, we again move left to right, calculating the multiplications and divisions, one at a time. Multiplication and division hold the same position in the ordering, so when you encounter one or the other at this step, do it. Once the multiplications and divisions have been calculated, we again move left to right, calculating the additions and subtractions, one at a time. Additions and subtractions hold the same position in the ordering, so when you encounter one or the other at this step, do it. (You may have previously learned the order of operations as PEMDAS, with parentheses first; we will add that aspect later on.) We'll explore this as we work an example.

Example 1 **Using Two Order of Operations**

Calculate $21 - 4 \times 13$.

Solution

There are no exponents in this expression, so the next operations to check are multiplication and division.

Step 1: Moving left to right, the first multiplication encountered is 4 multiplied by 13. We perform that operation first.

$$21 - 4 \times 13$$
$$= 21 - 52$$

Step 2: The only operation remaining is the subtraction.

$$21 - 52 = -31$$

.

So, $21 - 4 \times 13 = -31$.

Example 2 **Using Two Order of Operations**

Calculate 4×8^3.

Solution

Step 1: Moving left to right, we see there is an exponent. We calculate the exponent first.

$$4 \times 8^3 = 4 \times (8 \times 8 \times 8) = 4 \times 512$$

Step 2: The only operation remaining is the multiplication.

$$4 \times 512 = 2,048$$

So, $4 \times 8^3 = 2,048$.

Example 3 **Using Three Order of Operations**

Calculate $2 + 3^2 \times 4$.

Solution

Step 1: To calculate this, move left to right, and compute all the exponents first. The only exponent we see is the squaring of the 3, so that is calculated first.

$$2 + 3^2 \times 4 = 2 + (3 \times 3) \times 4 = 2 + 9 \times 4$$

Step 2: Since the exponents are all calculated, now calculate all the multiplications and divisions moving left to right. The only multiplication or division present is 9 times 4.

$$2 + 9 \times 4 = 2 + 36$$

Step 3: Moving left to right, perform the additions and subtractions. There is only one such operation, 2 plus 36.

$$2 + 36 = 38$$

So, $2 + 3^2 \times 4 = 38$.

Even if the expression being calculated gets more complicated, we perform the operations in the order: EMDAS.

VIDEO

Order of Operations 1

Example 4 **Using Eight Order of Operations**

Correctly apply the order of operations to compute the following:

$$4 - 25 \times \frac{6}{10} \times 3^2 + 7 \times 2^3$$

.

Solution

Step 1: To do so, calculate the exponents first, moving left to right. There are two occurrences of exponents in the expression, 3 squared and 2 cubed.

$$4 - 25 \times \frac{6}{10} \times 3^2 + 7 \times 2^3 = 4 - 25 \times \frac{6}{10} \times 9 + 7 \times 8$$

Step 2: Now that the exponents are calculated, perform the multiplication and division, moving left to right. The first is the product of 25 and 6.

$$4 - 25 \times \frac{6}{10} \times 9 + 7 \times 8$$
$$= 4 - \frac{150}{10} \times 9 + 7 \times 8$$

Step 3: Next is the 150 divided by 10.

$$4 - \frac{150}{10} \times 9 + 7 \times 8$$
$$= 4 - 15 \times 9 + 7 \times 8$$

Step 4: Next is 15 multiplied by 9.

$$4 - 15 \times 9 + 7 \times 8$$
$$= 4 - 135 + 7 \times 8$$

Step 5: Finally, multiply the 7 and 8.

$$4 - 135 + 7 \times 8$$
$$= 4 - 135 + 56$$

As all the multiplications and divisions have been calculated, the additions and subtractions are performed, moving left to right.

$$4 - 135 + 56$$
$$= -131 + 56$$
$$= -131 + 56$$
$$= -75$$

The computed value is -75.

VIDEO

Order of Operations 2

Example 5 **Using Six Order of Operations**

Correctly apply the rules for the order of operations to accurately compute the following:

$$10 - 3 \times \frac{5^3}{15} + \frac{56}{4}$$

.

Solution

Step 1: Calculate exponents first, moving left to right:

$$10 - 3 \times \frac{5^3}{15} + \frac{56}{4}$$
$$= 10 - 3 \times \frac{125}{15} + \frac{56}{4}$$

Step 2: Multiply and divide, moving left to right:

$$10 - 3 \times \frac{125}{15} + \frac{56}{4}$$
$$= 10 - \frac{375}{15} + \frac{56}{4}$$
$$= 10 - 25 + \frac{56}{4}$$
$$= 10 - 25 + 14$$

Step 3: Add and subtract, moving left to right:

$$10 - 25 + 14$$
$$= -15 + 14$$
$$= -1$$

Example 6 **Using Order of Operations**

Correctly apply the rules for the order of operations to accurately compute the following: $\frac{-8}{2} \times 3 - 9 \times \frac{2^4}{12} + 9 \times \frac{(-4)^2}{2^3}$.

Solution

Step 1: Calculate the exponents first, moving left to right:

$$\frac{-8}{2} \times 3 - 9 \times \frac{2^4}{12} + 9 \times \frac{(-4)^2}{2^3}$$
$$= \frac{-8}{2} \times 3 - 9 \times \frac{16}{12} + 9 \times \frac{12^2}{2^3}$$
$$= \frac{-8}{2} \times 3 - 9 \times \frac{16}{12} + 9 \times \frac{144}{2^3}$$
$$= \frac{-8}{2} \times 3 - 9 \times \frac{16}{12} + 9 \times \frac{144}{8}$$

Step 2: Multiply and divide, moving left to right:

$$= \frac{-8}{2} \times 3 - 9 \times \frac{16}{12} + 9 \times \frac{144}{8}$$
$$= (-4) \times 3 - 9 \times \frac{16}{12} + 9 \times \frac{144}{8}$$
$$= (-12) - 9 \times \frac{16}{12} + 9 \times \frac{144}{8}$$
$$= (-12) - \frac{144}{12} + 9 \times \frac{144}{8}$$
$$= (-12) - 12 + 9 \times \frac{144}{8}$$
$$= (-12) - 12 + 1,\frac{296}{8}$$
$$= (-12) - 12 + 162$$

Step 3: Add and subtract, moving left to right:

$$= (-12) - 12 + 162 = (-24) + 162 = 138$$

Using the Order of Operations Involving Grouping Symbols

We have examined how to use the order of operations, denoted by EMDAS, to correctly calculate expressions. However, there may be expressions where a multiplication should happen before an exponent, or a subtraction before a division. To indicate an operation should be performed out of order, the operation is placed inside parentheses. When parentheses are present, the operations inside the parentheses are performed first. Adding the parentheses to our list, we now have PEMDAS, as shown below.

P	Parentheses	
E	Exponents	
M/D	Multiplication and division	(division is just the multiplication by the reciprocal)
A/S	Addition and subtraction	(subtraction is just the addition of the negative)

As said previously, parentheses indicate that some operation or operations will be performed outside the standard order of operation rules. For instance, perhaps you want to multiply 4 and 7 before squaring. To indicate that the multiplication happens before the exponent, the multiplication is placed inside parentheses: $(4 \times 7)^2$.

This means operations inside the parentheses take precedence, or happen before other operations. Now, the first step in calculating arithmetic expressions using the order of operations is to perform operations inside parentheses first. Inside the parentheses, you follow the order of operation rules EMDAS.

Example 7 **Prioritizing Parentheses in the Order of Operations**

Correctly apply the rules for the order of operations to accurately compute the following:

$$(10 - 3) \times 5^3$$

.

Solution

Step 1: Perform all calculations within the parentheses before all other operations.

$$(10 - 3) \times 5^3 = 7 \times 5^3$$

Step 2: Since all parentheses have been cleared, move left to right, and compute all the exponents next.

$$7 \times 5^3 = 7 \times 125$$

Step 3: Perform all multiplications and divisions moving left to right.

$$7 \times 125 = 875$$

Be aware that there can be more than one set of parentheses, and parentheses within parentheses. When one set of parentheses is inside another set, do the innermost set first, and then work outward.

VIDEO

Order of Operations 3

Example 8 **Working Innermost Parentheses in the Order of Operations**

Correctly apply the rules for order of operations to accurately compute the following:

$$4 + 2 \times \frac{3^2 - (2+5)^2 \times 4}{3+8}$$

.

Solution

Step 1: Perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time. The innermost set of parentheses has the $2 + 5$ inside. The $3 + 8$ is in a separate set of parentheses, so that addition can occur at the same time as the $2 + 5$.

$$\begin{aligned} &4 + 2 \times \frac{3^2 - (2+5)^2 \times 4}{3+8} \\ &= 4 + 2 \times \frac{3^2 - (7)^2 \times 4}{11} \end{aligned}$$

Step 2: Now that those parentheses have been handled, move on to the next set of parentheses. Applying the order of operation rules inside that set of parentheses, the exponent is evaluated first, then the multiplication, and then the addition.

$$\begin{aligned} &4 + 2 \times \frac{3^2 - 7^2 \times 4}{11} \\ &= 4 + 2 \times \frac{3^2 - 49 \times 4}{11} \\ &= 4 + 2 \times \frac{-187}{11} \end{aligned}$$

Step 3: Since all parentheses have been cleared, apply the EMDAS rules to finish the calculation.

$$4 + 2 \times \frac{-187}{11} = 4 - \frac{374}{11} = 4 - 34 = -30$$

VIDEO

Order of Operations 4

Key Terms

- order of operations
- PEMDAS

Key Concepts

- Establishing shared rules on which arithmetic operations are calculated first is necessary. Without them, different people may find different values for the same expression.

- The highest precedence is with expressions in parentheses. This allows parts of an expression to be calculated in an order different than the basic order of operations.
- The lowest precedence is addition and subtraction, as they are the basis for all other calculations.
- Multiplication and division have precedence over addition and subtraction, as they are representations of repeated addition or subtraction.
- Exponents have precedence over multiplication and division, as they represent repeated multiplication and division.

Videos

- Order of Operations 1
- Order of Operations 2
- Order of Operations 3
- Order of Operations 4

3.4 Rational Numbers

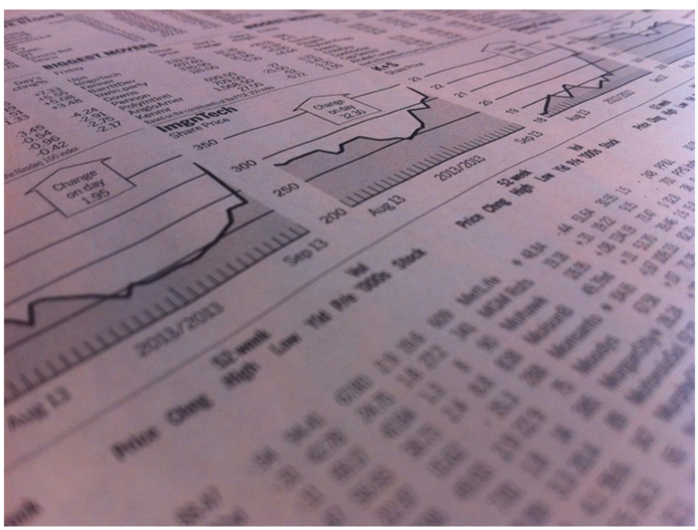


Figure 20: Stock gains and losses are often represented as percentages.

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are rational.
2. Simplify rational numbers and express in lowest terms.
3. Add and subtract rational numbers.
4. Convert between improper fractions and mixed numbers.

5. Convert rational numbers between decimal and fraction form.
6. Multiply and divide rational numbers.
7. Apply the order of operations to rational numbers to simplify expressions.
8. Apply density property of rational numbers.
9. Solve problems involving rational numbers.
10. Use fractions to convert between units.
11. Define and apply percent.
12. Solve problems using percent.

   We are often presented with percentages or fractions to explain how much of a population has a certain feature. For example, the 6-year graduation rate of college students at public institutions is 57.6%, or 72/125. That fraction may be unsettling. But without the context, the percentage is hard to judge. So how does that compare to private institutions? There, the 6-year graduation rate is 65.4%, or 327/500. Comparing the percentages is straightforward, but the fractions are harder to interpret due to different denominators. For more context, historical data could be found. One study reported that the 6-year graduation rate in 1995 was 56.4%. Comparing that historical number to the recent 6-year graduation rate at public institutions of 57.6% shows that there hasn't been much change in that rate.

Defining and Identifying Numbers That Are Rational

A **rational number** (called rational since it is a ratio) is just a fraction where the numerator is an integer and the denominator is a non-zero integer. As simple as that is, they can be represented in many ways. It should be noted here that any integer is a rational number. An integer, n, written as a fraction of two integers is $\frac{n}{1}$.

In its most basic representation, a rational number is an integer divided by a non-zero integer, such as $\frac{3}{12}$. Fractions may be used to represent parts of a whole. The denominator is the total number of parts to the object, and the numerator is how many of those parts are being used or selected. So, if a pizza is cut into 8 equal pieces, each piece is $\frac{1}{8}$ of the pizza. If you take three slices, you have $\frac{3}{8}$ of the pizza . Similarly, if in a group of 20 people, 5 are wearing hats, then $\frac{5}{20}$ of the people are wearing hats .

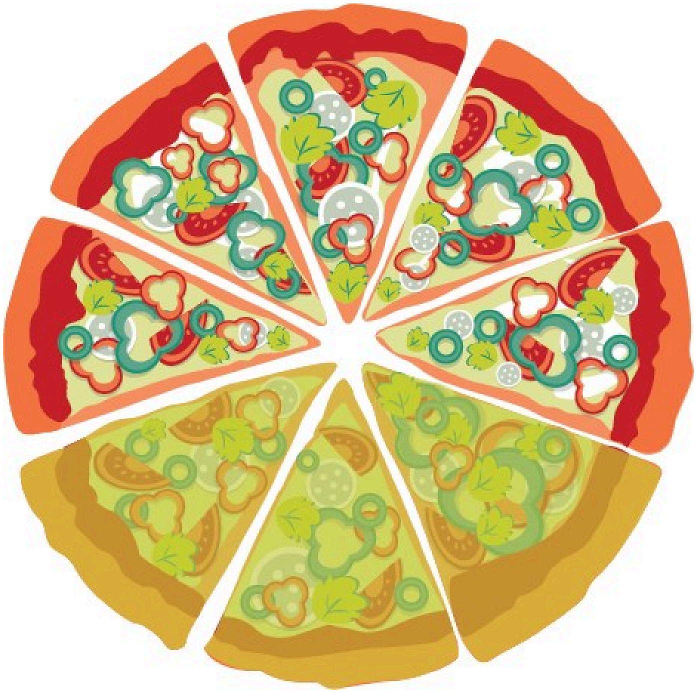


Figure 21: Pizza cut in 8 slices, with 3 slices highlighted

Figure 22: Group of 20 people, with 5 people wearing hats

Another representation of rational numbers is as a mixed number, such as $2\frac{5}{8}$. This represents a whole number (2 in this case), plus a fraction (the $\frac{5}{8}$).

Figure 23: Two whole pizzas and one partial pizza

Rational numbers may also be expressed in decimal form; for instance, as 1.34. When 1.34 is written, the decimal part, 0.34, represents the fraction $\frac{34}{100}$, and the number 1.34 is equal to $1\frac{34}{100}$. However, not all decimal representations are rational numbers.

A number written in decimal form where there is a last decimal digit (after a given decimal digit, all following decimal digits are 0) is a **terminating decimal**, as in 1.34 above. Alternately, any decimal numeral that, after a finite number of decimal digits, has digits equal to 0 for all digits following the last non-zero digit.

All numbers that can be expressed as a terminating decimal are rational. This comes from what the decimal represents. The decimal part is the fraction of the decimal part divided by the appropriate power of 10. That power of 10 is the number of decimal digits present, as for 0.34, with two decimal digits, being equal to $\frac{34}{100}$.

Another form that is a rational number is a decimal that repeats a pattern, such as 67.1313... When a rational number is expressed in decimal form and the decimal is a repeated pattern, we use special notation to designate the part that repeats. For example, if we have the repeating decimal 4.3636..., we write this as $4.\overline{36}$. The bar over the 36 indicates that the 36 repeats forever.

If the decimal representation of a number does not terminate or form a repeating decimal, that number is not a rational number.

One class of numbers that is not rational is the **square roots** of integers or rational numbers that are not **perfect squares**, such as $\sqrt{10}$ and $\sqrt{\frac{25}{6}}$. More generally, the number b is the square root of the number a if $a = b^2$. The notation for this is $b = \sqrt{a}$, where the symbol $\sqrt{}$ is the square root sign. An integer perfect square is any integer that can be written as the square of another integer. A rational perfect square is any rational number that can be written as a fraction of two integers that are perfect squares.

Sometimes you may be able to identify a perfect square from memory. Another process that may be used is to factor the number into the product of an integer with itself. Or a calculator (such as Desmos) may be used to find the square root of the number. If the calculator yields an integer, the original number was a perfect square.

TECH CHECK

Using Desmos to Find the Square Root of a Number

When Desmos is used, there is a tab at the bottom of the screen that opens the keyboard for Desmos. The keyboard is shown below. On the keyboard is the square root symbol $\left(\sqrt{}\right)$.

To find the square root of a number, click the square root key, and then type the number. Desmos will automatically display the value of the square root as you enter the number.

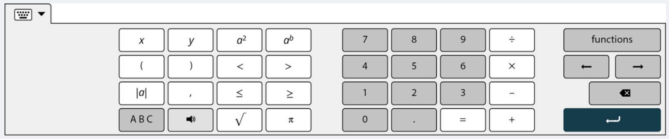


Figure 24: Desmos keyboard with square root key

Example 1 **Identifying Perfect Squares**

Which of the following are perfect squares?

1. 45
2. 144

Solution

1. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 45 results in $1 \times 45, 3 \times 15$, and 5×9. None of the pairs is a square, so 45 is not a perfect square. Using a calculator to find the square root of 45, we obtain 6.708 (rounded to three decimal places). Since this was not an integer, the original number was not a perfect square.
2. We could attempt to find the perfect square by factoring. Writing all the factor pairs of 144 results in $1 \times 144, 2 \times 72, 3 \times 48, 6 \times 24, 8 \times 18$, and 12×12. Since the last pair is an integer multiplied by itself, 144 is a perfect square. Alternately, using Desmos to find the square root of 144, we obtain 12. Since the square root of 144 is an integer, 144 is a perfect square.

VIDEO

Introduction to Fractions

Example 2 **Identifying Rational Numbers**

Determine which of the following are rational numbers:

1. $\sqrt{73}$
2. 4.556
3. $3\frac{1}{5}$
4. $\frac{41}{17}$
5. $5.\overline{64}$

Solution

1. Since 73 is not a perfect square, its square root is not a rational number. This can also be seen when a calculator is used. Entering $\sqrt{73}$ into a calculator results in 8.544003745317 (and then more decimal values after that). There is no repeated pattern, so this is not a rational number.
2. Since 4.556 is a decimal that terminates, this is a rational number.
3. $3\frac{1}{5}$ is a mixed number, so it is a rational number.
4. $\frac{41}{17}$ is an integer divided by an integer, so it is a rational number.
5. 5.646464... is a decimal that repeats a pattern, so it is a rational number.

Simplifying Rational Numbers and Expressing in Lowest Terms

A rational number is one way to express the division of two integers. As such, there may be multiple ways to express the same value with different rational numbers. For instance, $\frac{4}{5}$ and $\frac{12}{15}$ are the same value. If we enter them into a calculator, they both equal 0.8. Another way to understand this is to consider what it looks like in a figure when two fractions are equal.

In , we see that $\frac{3}{5}$ of the rectangle and $\frac{9}{15}$ of the rectangle are equal areas.

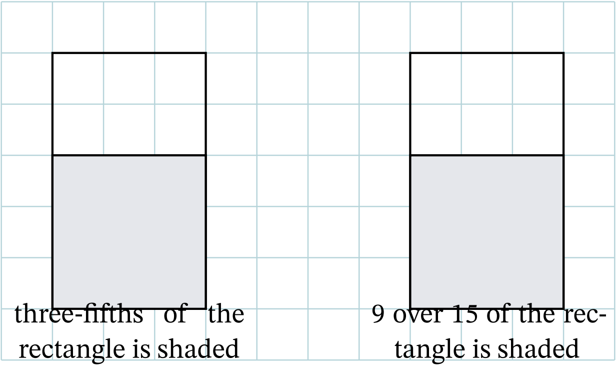


Figure 25: Two Rectangles with Equal Areas

They are the same proportion of the area of the rectangle. The left rectangle has 5 pieces, three of which are shaded. The right rectangle has 15 pieces, 9 of which are shaded. Each of the pieces of the left rectangle was divided equally into three pieces. This was a multiplication. The numerator describing the left rectangle was 3 but it becomes 3×3, or 9, as each piece was divided into three. Similarly, the denominator describing the left rectangle was 5, but became 5×3, or 15, as each piece was divided into 3. The fractions $\frac{3}{5}$ and $\frac{9}{15}$ are **equivalent** because they represent the same portion (often loosely referred to as equal).

This understanding of equivalent fractions is very useful for conceptualization, but it isn't practical, in general, for determining when two fractions are equivalent. Generally, to determine if the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent, we check to see that $a \times d = b \times c$. If those two products are equal, then the fractions are equal also.

Example 3 **Determining If Two Fractions Are Equivalent**

Determine if $\frac{12}{30}$ and $\frac{14}{35}$ are equivalent fractions.

Solution

Applying the definition, $a = 12, b = 30, c = 14$ and $d = 35$. So $a \times d = 12 \times 35 = 420$. Also, $b \times c = 30 \times 14 = 420$. Since these values are equal, the fractions are equivalent.

That $a \times d = b \times c$ indicates the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent is due to some algebra. One property of natural numbers, integers, and rational numbers (also irrational numbers) is that for any three numbers $a, b,$ and c with $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$. In other words, when two numbers are equal, then dividing both numbers by the same non-zero number, the two newly obtained numbers are also equal. We can apply that to $a \times d$ and $b \times c$, to show that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $a \times d = b \times c$.

If $a \times d = b \times c$, and $c \neq 0, d \neq 0$, we can divide both sides by and obtain the following: $\frac{a \times d}{c} = \frac{b \times c}{c}$. We can divide out the c on the right-hand side of the equation, resulting in $\frac{a \times d}{c} = b$. Similarly, we can divide both sides of the equation by d and obtain the following: $\frac{a \times d}{c \times d} = \frac{b}{d}$. We can divide out d the on the left-hand side of the equation, resulting in $\frac{a}{c} = \frac{b}{d}$. So, the rational numbers $\frac{a}{c}$ and $\frac{b}{d}$ are equivalent when $a \times d = b \times c$.

VIDEO

Equivalent Fractions

Recall that a **common divisor** or **common factor** of a set of integers is one that divides all the numbers of the set of numbers being considered. In a fraction, when the numerator and denominator have a common divisor, that common divisor can be **divided out**. This is often called **canceling the common factors** or, more colloquially, as **canceling**.

To show this, consider the fraction $\frac{36}{63}$. The numerator and denominator have the common factor 3. We can rewrite the fraction as $\frac{36}{63} = \frac{12 \times 3}{21 \times 3}$. The common divisor 3 is then divided out, or canceled, and we can write the fraction as $\frac{12 \times \cancel{3}}{21 \times \cancel{3}} = \frac{12}{21}$. The 3s have been crossed out to indicate they have been divided out. The process of dividing out two factors is also referred to as **reducing the fraction**.

If the numerator and denominator have no common positive divisors other than 1, then the rational number is in **lowest terms**.

The process of dividing out common divisors of the numerator and denominator of a fraction is called **reducing the fraction**.

One way to reduce a fraction to lowest terms is to determine the GCD of the numerator and denominator and divide out the GCD. Another way is to divide out common divisors until the numerator and denominator have no more common factors.

Example 4 **Reducing Fractions to Lowest Terms**

Express the following rational numbers in lowest terms:

1. $\frac{36}{48}$
2. $\frac{100}{250}$
3. $\frac{51}{136}$

Solution

1. One process to reduce $\frac{36}{48}$ to lowest terms is to identify the GCD of 36 and 48 and divide out the GCD. The GCD of 36 and 48 is 12. **Step 1:** We can then rewrite the numerator and denominator by factoring 12 from both.

$$\frac{36}{48} = \frac{12 \times 3}{12 \times 4}$$

Step 2: We can now divide out the 12s from the numerator and denominator.

$$\frac{36}{48} = \frac{\cancel{12} \times 3}{\cancel{12} \times 4} = \frac{3}{4}$$

So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

Alternately, you could identify a common factor, divide out that common factor, and repeat the process until the remaining fraction is in lowest terms.

Step 1: You may notice that 4 is a common factor of 36 and 48.

Step 2: Divide out the 4, as in $\frac{36}{48} = \frac{4\times 9}{4\times 12} = \frac{\cancel{4}\times 9}{\cancel{4}\times 12} = \frac{9}{12}$.

Step 3: Examining the 9 and 12, you identify 3 as a common factor and divide out the 3, as in $\frac{9}{12} = \frac{\cancel{3}\times 3}{\cancel{3}\times 4} = \frac{3}{4}$. The 3 and 4 have no common positive factors other than 1, so it is in lowest terms.

So, when $\frac{36}{48}$ is reduced to lowest terms, the result is $\frac{3}{4}$.

1. **Step 1:** To reduce $\frac{100}{250}$ to lowest terms, identify the GCD of 100 and 250. This GCD is 50. **Step 2:** We can then rewrite the numerator and denominator by factoring 50 from both.

$$\frac{100}{250} = \frac{50 \times 2}{50 \times 5}$$

.

Step 3: We can now divide out the 50s from the numerator and denominator.

$$\frac{100}{250} = \frac{\cancel{50} \times 2}{\cancel{50} \times 5} = \frac{2}{5}$$

So, when $\frac{100}{250}$ is reduced to lowest terms, the result is $\frac{2}{5}$.

1. **Step 1:** To reduce $\frac{51}{136}$ to lowest terms, identify the GCD of 51 and 136. This GCD is 17. **Step 2:** We can then rewrite the numerator and denominator by factoring 17 from both.

$$\frac{51}{136} = \frac{17 \times 3}{17 \times 8}$$

Step 3: We can now divide out the 17s from the numerator and denominator.

$$\frac{51}{136} = \frac{\cancel{17} \times 3}{\cancel{17} \times 8} = \frac{3}{8}$$

So, when $\frac{51}{136}$ is reduced to lowest terms, the result is $\frac{3}{8}$.

VIDEO

Reducing Fractions to Lowest Terms

TECH CHECK

Using Desmos to Find Lowest Terms

Desmos is a free online calculator. Desmos supports reducing fractions to lowest terms. When a fraction is entered, Desmos immediately calculates the decimal representation of the fraction. However, to the left of the fraction, there is a button that, when clicked, shows the fraction in reduced form.

VIDEO

Using Desmos to Reduce a Fraction

Adding and Subtracting Rational Numbers

Adding or subtracting rational numbers can be done with a calculator, which often returns a decimal representation, or by finding a common denominator for the rational numbers being added or subtracted.

TECH CHECK

Using Desmos to Add Rational Numbers in Fractional Form

To create a fraction in Desmos, enter the numerator, then use the division key (/) on your keyboard, and then enter the denominator. The fraction is then entered. Then click the right arrow key to exit the denominator of the fraction. Next, enter the arithmetic operation (+ or –). Then enter the next fraction. The answer is displayed dynamically (calculates as you enter). To change the Desmos result from decimal form to fractional form, use the fraction button on the left of the line that contains the calculation:

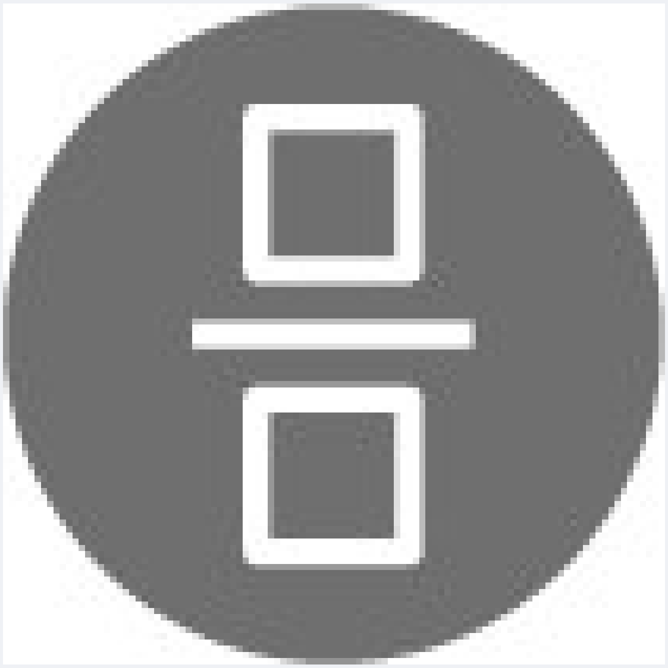


Figure 26: Fraction button on the Desmos keyboard

Example 5 **Adding Rational Numbers Using Desmos**

Calculate $\frac{23}{42} + \frac{9}{56}$ using Desmos.

Solution

Enter $\frac{23}{42} + \frac{9}{56}$ in Desmos. The result is displayed as 0.70833333333 (which is $0.708\overline{3}$). Clicking the fraction button to the left on the calculation line yields $\frac{17}{24}$.

Performing addition and subtraction without a calculator may be more involved. When the two rational numbers have a common denominator, then adding or subtracting the two numbers is straightforward. Add or subtract the numerators, and then place that value in the numerator and the common denominator in the denominator. Symbolically, we write this as $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$. This can be seen in the , which shows $\frac{3}{20} + \frac{4}{20} = \frac{7}{20}$.

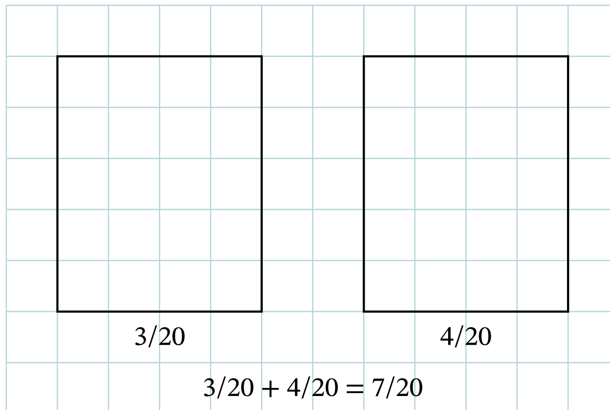


Figure 27: Partially Shaded Rectangle

It is customary to then write the result in lowest terms.

FORMULA

If c is a non-zero integer, then $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$.

Example 6 **Adding Rational Numbers with the Same Denominator**

Calculate $\frac{13}{28} + \frac{7}{28}$.

Solution

Since the rational numbers have the same denominator, we perform the addition of the numerators, $13 + 7$, and then place the result in the numerator and the common denominator, 28, in the denominator. $\frac{13}{28} + \frac{7}{28} = \frac{13+7}{28} = \frac{20}{28}$

Once we have that result, reduce to lowest terms, which gives $\frac{20}{28} = \frac{4 \times 5}{4 \times 7} = \frac{\cancel{4} \times 5}{\cancel{4} \times 7} = \frac{5}{7}$.

Example 7 **Subtracting Rational Numbers with the Same Denominator**

Calculate $\frac{45}{136} - \frac{17}{136}$.

Solution

Since the rational numbers have the same denominator, we perform the subtraction of the numerators, $45 - 17$, and then place the result in the numerator and the common denominator, 136, in the denominator. $\frac{45}{136} - \frac{17}{136} = \frac{45-17}{136} = \frac{28}{136}$

Once we have that result, reduce to lowest terms, this gives $\frac{28}{136} = \frac{4 \times 7}{4 \times 34} = \frac{\cancel{4} \times 7}{\cancel{4} \times 34} = \frac{7}{34}$.

When the rational numbers do not have common denominators, then we have to transform the rational numbers so that they do have common denominators. The common denominator that reduces work later in the problem is the LCM of the numerator and denominator. When adding or subtracting the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we perform the following steps.

Step 1: Find $LCM\,(b, d)$.

Step 2: Calculate $n = \frac{LCM(b,d)}{b}$ and $m = \frac{LCM(b,d)}{d}$.

Step 3: Multiply the numerator and denominator of $\frac{a}{b}$ by n, yielding $\frac{a \times n}{b \times n}$.

Step 4: Multiply the numerator and denominator of $\frac{c}{d}$ by m, yielding $\frac{c \times m}{d \times m}$.

Step 5: Add or subtract the rational numbers from Steps 3 and 4, since they now have the common denominators.

You should be aware that the common denominator is $LCM(b,d)$. For the first denominator, we have $b \times n = b \times \frac{LCM(b,d)}{b} = LCM(b,d)$, since we multiply and divide $LCM(b,d)$ by the same number. For the same reason, $d \times m = d \times \frac{LCM(b,d)}{b} = LCM(b,d)$.

Example 8 **Adding Rational Numbers with Unequal Denominators**

Calculate $\frac{11}{18} + \frac{2}{15}$.

Solution

The denominators of the fractions are 18 and 15, so we label $b = 18$ and $d = 15$.

Step 1: Find LCM(18,15). This is 90.

Step 2: Calculate n and m. $n = \frac{90}{18} = 5$ and $m = \frac{90}{15} = 6$.

Step 3: Multiplying the numerator and denominator of $\frac{11}{18}$ by $n = 5$ yields $\frac{11 \times 5}{18 \times 5} = \frac{55}{90}$.

Step 4: Multiply the numerator and denominator of $\frac{2}{15}$ by $m = 6$ yields $\frac{2 \times 6}{15 \times 6} = \frac{12}{90}$.

Step 5: Now we add the values from Steps 3 and 4: $\frac{55}{90} + \frac{12}{90} = \frac{67}{90}$.

This is in lowest terms, so we have found that $\frac{11}{18} + \frac{2}{15} = \frac{67}{90}$.

Example 9 **Subtracting Rational Numbers with Unequal Denominators**

Calculate $\frac{14}{25} - \frac{9}{70}$.

Solution

The denominators of the fractions are 25 and 70, so we label $b = 25$ and $d = 70$.

Step 1: Find LCM(25,70). This is 350.

Step 2: Calculate n and m: $n = \frac{350}{25} = 14$ and $m = \frac{350}{70} = 5$.

Step 3: Multiplying the numerator and denominator of $\frac{14}{25}$ by $n = 14$ yields $\frac{14 \times 14}{25 \times 14} = \frac{196}{350}$.

Step 4: Multiplying the numerator and denominator of $\frac{9}{70}$ by $m = 5$ yields $\frac{9 \times 5}{70 \times 5} = \frac{45}{350}$.

Step 5: Now we subtract the value from Step 4 from the value in Step 3: $\frac{196}{350} - \frac{45}{350} = \frac{151}{350}$.

This is in lowest terms, so we have found that $\frac{14}{25} - \frac{9}{70} = \frac{151}{350}$.

VIDEO

Adding and Subtracting Fractions with Different Denominators

Converting Between Improper Fractions and Mixed Numbers

One way to visualize a fraction is as parts of a whole, as in $\frac{5}{12}$ of a pizza. But when the numerator is larger than the denominator, as in $\frac{23}{12}$, then the idea of parts of a whole seems not to make sense. Such a fraction is an **improper fraction.** That kind of fraction could be written as an integer plus a fraction, which is a **mixed number**. The fraction $\frac{23}{12}$ rewritten as a mixed number would be $1\frac{11}{12}$. Arithmetically, $1\frac{11}{12}$ is equivalent to $1 + \frac{11}{12}$, which is read as "one and 11 twelfths."

Improper fractions can be rewritten as mixed numbers using division and remainders. To find the mixed number representation of an improper fraction, divide the numerator by the denominator. The quotient is the integer part, and the remainder becomes the numerator of the remaining fraction.

Example 10 **Rewriting an Improper Fraction as a Mixed Number**

Rewrite $\frac{48}{13}$ as a mixed number.

Solution

When 48 is divided by 13, the result is 3 with a remainder of 9. So, we can rewrite $\frac{48}{13}$ as $3\frac{9}{13}$.

VIDEO

Converting an Improper Fraction to a Mixed Number Using Desmos

Similarly, we can convert a mixed number into an improper fraction. To do so, first convert the whole number part to a fraction by writing the whole number as itself divided by 1, and then add the two fractions.

Alternately, we can multiply the whole number part and the denominator of the fractional part. Next, add that product to the numerator. Finally, express the number as that product divided by the denominator.

Example 11 **Rewriting a Mixed Number as an Improper Fraction**

Rewrite $5\frac{4}{9}$ as an improper fraction.

Solution

Step 1: Multiply the integer part, 5, by the denominator, 9, which gives $5 \times 9 = 45$.

Step 2: Add that product to the numerator, which gives $45 + 4 = 49$.

Step 3: Write the number as the sum, 49, divided by the denominator, 9, which gives $\frac{49}{9}$.

TECH CHECK

Using Desmos to Rewrite a Mixed Number as an Improper Fraction

Desmos can be used to convert from a mixed number to an improper fraction. To do so, we use the idea that a mixed number, such as $5\frac{6}{11}$, is another way to represent $5 + \frac{6}{11}$. If $5 + \frac{6}{11}$ is entered in Desmos, the result is the decimal form of the number. However, clicking the fraction button to the left will convert the decimal to an improper fraction, $\frac{61}{11}$. As an added bonus, Desmos will automatically reduce the fraction to lowest terms.

Converting Rational Numbers Between Decimal and Fraction Forms

Understanding what decimals represent is needed before addressing conversions between the fractional form of a number and its **decimal form**, or writing a number in **decimal notation**. The decimal number 4.557 is equal to $4\frac{557}{1{,}000}$. The decimal portion, .557, is 557 divided by 1,000. To write any decimal portion of a number expressed as a terminating decimal, divide the decimal number by 10 raised to the power equal to the number of decimal digits. Since there were three decimal digits in 4.557, we divided 557 by $10^3 = 1000$.

Decimal representations may be very long. It is convenient to **round off** the decimal form of the number to a certain number of decimal digits. To round off the decimal form of a number to n (decimal) digits, examine the $(n + 1)$st decimal digit. If that digit is 0, 1, 2, 3, or 4, the number is rounded off by writing the number to the nth decimal digit and no further. If the $(n + 1)$st decimal digit is 5, 6, 7, 8, or 9, the number is rounded off by writing the number to the nth digit, then replacing the nth digit by one more than the nth digit.

Example 12 **Rounding Off a Number in Decimal Form to Three Digits**

Round 5.67849 to three decimal digits.

Solution

The third decimal digit is 8. The digit following the 8 is 4. When the digit is 4, we write the number only to the third digit. So, 5.67849 rounded off to three decimal places is 5.678.

Example 13 **Rounding Off a Number in Decimal Form to Four Digits**

Round 45.11475 to four decimal digits.

Solution

The fourth decimal digit is 7. The digit following the 7 is 5. When the digit is 5, we write the number only to the fourth decimal digit, 45.1147. We then replace the fourth decimal digit by one more than the fourth digit, which yields 45.1148. So, 45.11475 rounded off to four decimal places is 45.1148.

To convert a rational number in fraction form to decimal form, use your calculator to perform the division.

Example 14 **Converting a Rational Number in Fraction Form into Decimal Form**

Convert $\frac{47}{25}$ into decimal form.

Solution

Using a calculator to divide 47 by 25, the result is 1.88.

Converting a terminating decimal to the fractional form may be done in the following way:

Step 1: Count the number of digits in the decimal part of the number, labeled n.

Step 2: Raise 10 to the nth power.

Step 3: Rewrite the number without the decimal.

Step 4: The fractional form is the number from Step 3 divided by the result from Step 2.

This process works due to what decimals represent and how we work with mixed numbers. For example, we could convert the number 7.4536 to fractional from. The decimal part of the number, the .4536 part of 7.4536, has four digits. By the definition of decimal notation, the decimal portion represents $\frac{4{,}536}{10^4} = \frac{4{,}536}{10{,}000}$. The decimal number 7.4536 is equal to the improper fraction $7\frac{4{,}536}{10{,}000}$. Adding those to fractions yields $\frac{74{,}536}{10{,}000}$.

Example 15 **Converting from Decimal Form to Fraction Form with Terminating Decimals**

Convert 3.2117 to fraction form.

Solution

Step 1: There are four digits after the decimal point, so $n = 4$.

Step 2: Raise 10 to the fourth power, $10^4 = 10,000$.

Step 3: When we remove the decimal point, we have 32,117.

Step 4: The fraction has as its numerator the result from Step 3 and as its denominator the result of Step 2, which is the fraction $\frac{32{,}117}{10{,}000}$.

The process is different when converting from the decimal form of a rational number into fraction form when the decimal form is a repeating decimal. This process is not covered in this text.

Multiplying and Dividing Rational Numbers

Multiplying rational numbers is less complicated than adding or subtracting rational numbers, as there is no need to find common denominators. To multiply rational numbers, multiply the numerators, then multiply the denominators, and write the numerator product divided by the denominator product. Symbolically, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. As always, rational numbers should be reduced to lowest terms.

FORMULA

If b and d are non-zero integers, then $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

Example 16 **Multiplying Rational Numbers**

Calculate $\frac{12}{25} \times \frac{10}{21}$.

Solution

Multiply the numerators and place that in the numerator, and then multiply the denominators and place that in the denominator.

$$\frac{12}{25} \times \frac{10}{21} = \frac{12 \times 10}{25 \times 21} = \frac{120}{525}$$

This is not in lowest terms, so this needs to be reduced. The GCD of 120 and 525 is 15.

$$\frac{120}{525} = \frac{15 \times 8}{15 \times 35} = \frac{8}{35}$$

VIDEO

Multiplying Fractions

As with multiplication, division of rational numbers can be done using a calculator.

Example 17 **Dividing Decimals with a Calculator**

Calculate 3.45 ÷ 2.341 using a calculator. Round to three decimal places if necessary.

Solution

Using a calculator, we obtain 1.473729175565997. Rounding to three decimal places we have 1.474.

Before discussing division of fractions without a calculator, we should look at the reciprocal of a number. The **reciprocal** of a number is 1 divided by the number. For a fraction, the reciprocal is the fraction formed by switching the numerator and denominator. For the fraction $\frac{a}{b}$, the reciprocal is $\frac{b}{a}$. An important feature for a number and its reciprocal is that their product is 1.

When dividing two fractions by hand, find the reciprocal of the **divisor** (the number that is being divided into the other number). Next, replace the divisor by its reciprocal and change the division into multiplication. Then, perform the multiplication. Symbolically, $\frac{b}{a} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$. As before, reduce to lowest terms.

FORMULA

If b, c and d are non-zero integers, then $\frac{b}{a} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$.

Example 18 **Dividing Rational Numbers**

1. Calculate $\frac{4}{21} \div \frac{6}{35}$.
2. Calculate $\frac{1}{8} \div \frac{5}{28}$.

Solution

1. **Step 1:** Find the reciprocal of the number being divided by $\frac{6}{35}$. The reciprocal of that is $\frac{35}{6}$.

Step 2: Multiply the first fraction by that reciprocal.

$$\frac{4}{21} \div \frac{6}{35} = \frac{4}{21} \times \frac{35}{6} = \frac{140}{126}$$

The answer, $\frac{140}{126}$ is not in lowest terms. The GCD of 140 and 126 is 14. Factoring and canceling gives $\frac{140}{126} = \frac{14 \times 10}{14 \times 9} = \frac{10}{9}$.

1. **Step 1:** Find the reciprocal of the number being divided by, which is $\frac{5}{28}$. The reciprocal of that is $\frac{28}{5}$.

Step 2: Multiply the first fraction by that reciprocal: $\frac{1}{8} \div \frac{5}{28} = \frac{1}{8} \times \frac{28}{5} = \frac{28}{40}$

The answer, $\frac{28}{40}$, is not in lowest reduced form. The GCD of 28 and 40 is 4. Factoring and canceling gives $\frac{28}{40} = \frac{4 \times 7}{4 \times 10} = \frac{7}{10}$.

VIDEO

Dividing Fractions

Applying the Order of Operations to Simplify Expressions

The order of operations for rational numbers is the same as for integers, as discussed in Order of Operations. The order of operations makes it easier for anyone to correctly calculate and represent. The order follows the well-known acronym PEMDAS:

P	Parentheses
E	Exponents
M/D	Multiplication and division
A/S	Addition and subtraction

The first step in calculating using the order of operations is to perform operations inside the parentheses. Moving down the list, next perform all exponent operations moving from left to right. Next (left to right once more), perform all multiplications and divisions. Finally, perform the additions and subtractions.

Example 19 **Applying the Order of Operations with Rational Numbers**

Correctly apply the rules for the order of operations to accurately compute $\left(\frac{5}{7} - \frac{2}{7}\right) \times 2^3$.

Solution

Step 1: To calculate this, perform all calculations within the parentheses before other operations.

$$\left(\frac{5}{7} - \frac{2}{7}\right) \times 2^3 = \left(\frac{3}{7}\right) \times 2^3$$

Step 2: Since all parentheses have been cleared, we move left to right, and compute all the exponents next.

$$\left(\frac{3}{7}\right) \times 2^3 = \left(\frac{3}{7}\right) \times 8$$

Step 3: Now, perform all multiplications and divisions, moving left to right.

$$\left(\frac{3}{7}\right) \times 8 = \frac{24}{7}$$

Example 20 **Applying the Order of Operations with Rational Numbers**

Correctly apply the rules for the order of operations to accurately compute $4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{2}{3} + 5\right)\right)^2$.

Solution

To calculate this, perform all calculations within the parentheses before other operations. Evaluate the innermost parentheses first. We can work separate parentheses expressions at the same time.

Step 1: The innermost parentheses contain $\frac{2}{3} + 5$. Calculate that first, dividing after finding the common denominator.

$$\begin{aligned} &4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{2}{3} + 5\right)\right)^2 \\ &= 4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{2}{3} + \frac{5}{1}\right)\right)^2 \\ &= 4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{2}{3} + \frac{15}{3}\right)\right)^2 \\ &= 4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{17}{3}\right)\right)^2 \end{aligned}$$

Step 2: Calculate the exponent in the parentheses, $\left(\frac{5}{9}\right)^2$.

$$\begin{aligned} &4 + \frac{2}{3} \div \left(\left(\frac{5}{9}\right)^2 - \left(\frac{17}{3}\right)\right)^2 \\ &= 4 + \frac{2}{3} \div \left(\left(\frac{25}{81}\right) - \left(\frac{17}{3}\right)\right)^2 \end{aligned}$$

Step 3: Subtract inside the parentheses is done, using a common denominator.

$$4+\frac{2}{3}\div\left(\left(\frac{25}{81}\right)-\left(\frac{17}{3}\right)\right)^2$$

$$4+\frac{2}{3}\div\left(\left(\frac{25}{81}\right)-\left(\frac{17\times 27}{3\times 27}\right)\right)^2$$

$$4+\frac{2}{3}\div\left(\left(\frac{25}{81}\right)-\left(\frac{459}{81}\right)\right)^2$$

$$4+\frac{2}{3}\div\left(\left(\frac{-434}{81}\right)\right)^2$$

Step 4: At this point, evaluate the exponent and divide.

$$4+\frac{2}{3}\div\left(\left(\frac{-434}{81}\right)\right)^2$$

$$4+\frac{2}{3}\div\left(\frac{188{,}356}{6{,}561}\right)$$

$$=4+\frac{2}{3}\times\left(\frac{6{,}561}{188{,}356}\right)$$

$$=4+\frac{2{,}187}{94{,}178}$$

Step 5: Add.

$$4+\frac{2{,}187}{94{,}178}$$

$$=\frac{378{,}899}{94{,}178}$$

Had this been done on a calculator, the decimal form of the answer would be 4.0232 (rounded to four decimal places).

VIDEO

Order of Operations Using Fractions

Applying the Density Property of Rational Numbers

Between any two rational numbers, there is another rational number. This is called the **density property** of the rational numbers.

Finding a rational number between any two rational numbers is very straightforward.

Step 1: Add the two rational numbers.

Step 2: Divide that result by 2.

The result is always a rational number. This follows what we know about rational numbers. If two fractions are added, then the result is a fraction. Also, when a fraction is divided by a fraction (and 2 is a fraction), then we get another fraction. This two-step process will give a rational number, provided the first two numbers were rational.

Example 21 **Applying the Density Property of Rational Numbers**

Demonstrate the density property of rational numbers by finding a rational number between $\frac{4}{11}$ and $\frac{7}{12}$.

Solution

To find a rational number between $\frac{4}{11}$ and $\frac{7}{12}$:

Step 1: Add the fractions.

$$\frac{4}{11}+\frac{7}{12}=\frac{4\times 12}{11\times 12}+\frac{7\times 11}{12\times 11}=\frac{48}{132}+\frac{77}{132}=\frac{125}{132}$$

Step 2: Divide the result by 2. Recall that to divide by 2, you multiply by the reciprocal of 2. The reciprocal of 2 is $\frac{1}{2}$, as seen below.

$$\frac{125}{132}\div 2=\frac{125}{132}\times\frac{1}{2}=\frac{125}{264}$$

So, one rational number between $\frac{4}{11}$ and $\frac{7}{12}$ is $\frac{125}{264}$.

We could check that the number we found is between the other two by finding the decimal representation of the numbers. Using a calculator, the decimal representations of the rational numbers are 0.363636…, 0.473484848…, and 0.5833333…. Here it is clear that $\frac{125}{264}$ is between $\frac{4}{11}$ and $\frac{7}{12}$.

Solving Problems Involving Rational Numbers

Rational numbers are used in many situations, sometimes to express a portion of a whole, other times as an expression of a ratio between two quantities. For the sciences, converting between units is done using rational numbers, as when converting between gallons and cubic inches. In chemistry, mixing a solution with a given concentration of a chemical per unit volume can be solved with rational numbers. In demographics, rational numbers are used to describe the distribution of the population. In dietetics, rational numbers are used to express the appropriate amount of a given ingredient to include in a recipe. As discussed, the application of rational numbers crosses many disciplines.

Example 22 **Mixing Soil for Vegetables**

James is mixing soil for a raised garden, in which he plans to grow a variety of vegetables. For the soil to be suitable, he determines that $\frac{2}{5}$ of the soil can be topsoil, but $\frac{2}{5}$ needs to be peat moss and $\frac{1}{5}$ has to be compost. To fill the raised garden bed with 60 cubic feet of soil, how much of each component does James need to use?

Solution

In this example, we know the proportion of each component to mix, and we know the total amount of the mix we need. In this kind of situation, we need to determine the appropriate amount of each component to include in the mixture. For each component of the mixture, multiply 60 cubic feet, which is the total volume of the mixture we want, by the fraction required of the component.

Step 1: The required fraction of topsoil is $\frac{2}{5}$, so James needs $60 \times \frac{2}{5}$ cubic feet of topsoil. Performing the multiplication, James needs $60 \times \frac{2}{5} = \frac{120}{5} = 24$ (found by treating the fraction as division, and 120 divided by 5 is 24) cubic feet of topsoil.

Step 2: The required fraction of peat moss is also $\frac{2}{5}$, so he also needs $60 \times \frac{2}{5}$ cubic feet, or $60 \times \frac{2}{5} = \frac{120}{5} = 24$ cubic feet of peat moss.

Step 3: The required fraction of compost is $\frac{1}{5}$. For the compost, he needs $60 \times \frac{1}{5} = \frac{60}{5} = 12$ cubic feet.

Example 23 **Determining the Number of Specialty Pizzas**

At Bella's Pizza, one-third of the pizzas that are ordered are one of their specialty varieties. If there are 273 pizzas ordered, how many were specialty pizzas?

Solution

One-third of the whole are specialty pizzas, so we need one-third of 273, which gives $\frac{1}{3} \times 273 = \frac{273}{3} = 91$, found by dividing 273 by 3. So, 91 of the pizzas that were ordered were specialty pizzas.

VIDEO

Finding a Fraction of a Total

Using Fractions to Convert Between Units

A common application of fractions is called **unit conversion**, or **converting units**, which is the process of changing from the units used in making a measurement to different units of measurement.

For instance, 1 inch is (approximately) equal to 2.54 cm. To convert between units, the two equivalent values are made into a fraction. To convert from the first type of unit to the second type, the fraction has the second unit as the numerator, and the first unit as the denominator.

From the inches and centimeters example, to change from inches to centimeters, we use the fraction $\frac{2.54 \text{ cm}}{1 \text{ in}}$. If, on the other hand, we wanted to convert from centimeters to inches, we'd use the fraction $\frac{1 \text{ in}}{2.54 \text{ cm}}$. This fraction is multiplied by the number of units of the type you are converting *from*, which means the units of the denominator are the same as the units being multiplied.

Example 24 **Converting Liters to Gallons**

It is known that 1 liter (L) is 0.264172 gallons (gal). Use this to convert 14 liters into gallons.

Solution

We know that 1 liter = 0.264172 gal. Since we are converting from liters, when we create the fraction we use, make sure the liter part of the equivalence is in the denominator. So, to convert the 14 liters to gallons, we multiply 14 by $\frac{1 \text{ gal}}{0.264172 \frac{\text{gal}}{1} \text{ liter}}$. Notice the gallon part is in the numerator since we're converting *to* gallons, and the liter part is in the denominator since we are converting *from* liters. Performing this and rounding to three decimal places, we find that 14 liters is 14 liter $\times \frac{0.264172 \text{ gal}}{1 \text{ liter}} = 3.69841$ gal.

Example 25 **Converting Centimeters to Inches**

It is known that 1 inch is 2.54 centimeters. Use this to convert 100 centimeters into inches.

Solution

We know that 1 inch = 2.54 cm. Since we are converting from centimeters, when we create the fraction we use, make sure the centimeter part of the equivalence is in the denominator, $\frac{1 \text{ in}}{2.54 \text{ cm}}$. To convert the 100 cm to inches, multiply 100 by $\frac{1 \text{ in}}{2.54 \text{ cm}}$. Notice the inch part is in the numerator since we're converting *to* inches, and the centimeter part is in the denominator since we are converting *from* centimeters. Performing this and rounding to three decimal places, we obtain 100 cm $\times \frac{1 \text{ in}}{2.54 \text{ cm}} = 39.370$ in. This means 100 cm equals 39.370 in.

VIDEO

Converting Units

Defining and Applying Percent

A **percent** is a specific rational number and is literally per 100. n percent, denoted $n\%$, is the fraction $\frac{n}{100}$.

Example 26 **Rewriting a Percentage as a Fraction**

Rewrite the following as fractions:

1. 31%
2. 93%

Solution

1. Using the definition and $n = 31$, 31% in fraction form is $\frac{31}{100}$.
2. Using the definition and $n = 93$, 93% in fraction form is $\frac{93}{100}$.

Example 27 **Rewriting a Percentage as a Decimal**

Rewrite the following percentages in decimal form:

1. 54%

2. 83%

Solution

1. Using the definition and $n = 54$, 54% in fraction form is $\frac{54}{100}$. Dividing a number by 100 moves the decimal two places to the left; 54% in decimal form is then 0.54.
2. Using the definition and $n = 83$, 83% in fraction form is $\frac{83}{100}$. Dividing a number by 100 moves the decimal two places to the left; 83% in decimal form is then 0.83.

You should notice that you can simply move the decimal two places to the left without using the fractional definition of percent.

Percent is used to indicate a fraction of a total. If we want to find 30% of 90, we would perform a multiplication, with 30% written in either decimal form or fractional form. The 90 is the **total**, 30 is the **percentage**, and 27 (which is 0.30×90) is the **percentage of the total**.

FORMULA

$n\%$ of x items is $\frac{n}{100} \times x$. The x is referred to as the **total**, the n is referred to as the **percent** or **percentage**, and the value obtained from $\frac{n}{100} \times x$ is the **part** of the total and is also referred to as the **percentage of the total**.

Example 28 **Finding a Percentage of a Total**

1. Determine 40% of 300.
2. Determine 64% of 190.

Solution

1. The total is 300, and the percentage is 40. Using the decimal form of 40% and multiplying we obtain $0.40 \times 300 = 120$.
2. The total is 190, and the percentage is 64. Using the decimal form of 64% and multiplying we obtain $0.64 \times 190 = 121.6$.

In the previous situation, we knew the total and we found the percentage of the total. It may be that we know the percentage of the total, and we know the percent, but we don't know the total. To find the total if we know the percentage of the total, use the following formula.

FORMULA

If we know that $n\%$ of the total is x, then the total is $\frac{100 \times x}{n}$.

Example 29 **Finding the Total When the Percentage and Percentage of the Total Are Known**

1. What is the total if 28% of the total is 140?

2. What is the total if 6% of the total is 91?

Solution

1. 28 is the percentage, so $n = 28$. 28% of the total is 140, so $x = 140$. Using those we find that the total was $\frac{100 \times 140}{28} = 500$.
2. 6 is the percentage, so $n = 6$. 6% of the total is 91, so $x = 91$. Using those we find that the total was $\frac{100 \times 91}{6} = 1,516.6$.

The percentage can be found if the total and the percentage of the total is known. If you know the total, and the percentage of the total, first divide the part by the total. Move the decimal two places to the right and append the symbol %. The percentage may be found using the following formula.

FORMULA

The percentage, n, of b that is a is $\frac{a}{b} \times 100\%$.

Example 30 **Finding the Percentage When the Total and Percentage of the Total Are Known**

Find the percentage in the following:

1. Total is 300, percentage of the total is 60.
2. Total is 440, percentage of the total is 176.

Solution

1. The total is 300; the percentage of the total is 60. Calculating yields 0.2. Moving the decimal two places to the right gives 20. Appending the percentage to this number results in 20%. So, 60 is 20% of 300.
2. The total is 440; the percentage of the total is 176. Calculating yields 0.4. Moving the decimal two places to the right gives 40. Appending the percentage to this number results in 40%. So, 176 is 40% of 440.

Solve Problems Using Percent

In the media, in research, and in casual conversation percentages are used frequently to express proportions. Understanding how to use percent is vital to consuming media and understanding numbers. Solving problems using percentages comes down to identifying which of the three components of a percentage you are given, the total, the percentage, or the percentage of the total. If you have two of those components, you can find the third using the methods outlined previously.

Example 31 **Percentage of Students Who Are Sleep Deprived**

A study revealed that 70% of students suffer from sleep deprivation, defined to be sleeping less than 8 hours per night. If the survey had 400 participants, how many of those participants had less than 8 hours of sleep per night?

Solution

The percentage of interest is 70%. The total number of students is 400. With that, we can find how many were in the percentage of the total, or, how many were sleep deprived. Applying the formula from above, the number who were sleep deprived was $0.70 \times 400 = 280$ 280 students on the study were sleep deprived.

Example 32 **Amazon Prime Subscribers**

There are 126 million users who are U.S. Amazon Prime subscribers. If there are 328.2 million residents in the United States, what percentage of U.S. residents are Amazon Prime subscribers?

Solution

We are asked to find the percentage. To do so, we divide the percentage of the total, which is 126 million, by the total, which is 328.2 million. Performing this division and rounding to three decimal places yields $\frac{126}{328.2} = 0.384$. The decimal is moved to the right by two places, and a % sign is appended to the end. Doing this shows us that 38.4% of U.S. residents are Amazon Prime subscribers.

Example 33 **Finding the Percentage When the Total and Percentage of the Total Are Known**

Evander plays on the basketball team at their university and 73% of the athletes at their university receive some sort of scholarship for attending. If they know 219 of the student-athletes receive some sort of scholarship, how many student-athletes are at the university?

Solution

We need to find the total number of student-athletes at Evander's university.

Step 1: Identify what we know. We know the percentage of students who receive some sort of scholarship, 73%. We also know the number of athletes that form the part of the whole, or 219 student-athletes.

Step 2: To find the total number of student-athletes, use $\frac{100 \times x}{n}$, with $x = 219$ and $n = 73$. Calculating with those values yields $\frac{100 \times 219}{73} = 300$.

So, there are 300 total student-athletes at Evander's university

Key Terms

- density property of rational numbers
- improper fraction
- lowest terms

- mixed number
- rational number
- repeating decimal
- terminating decimal

Key Concepts

- Rational numbers are fractions of integers, and can always be written as an integer divided by an integer.
- The numerator and denominator of a fraction may have common factors. In such cases, the fraction can be reduced by canceling common factors. When the numerator and denominator of a fraction have no common factors, the fraction is said to be reduced.
- An improper fraction is one with a numerator larger than the denominator. Such a fraction can be rewritten as an integer plus a proper fraction. This is called a mixed number.
- Using division and remainder, an improper fraction may be written as a mixed number.
- A mixed number can be converted to an improper fraction by reversing the process for changing an improper fraction to a mixed number.
- The arithmetic operations or addition, subtraction, multiplication and division can all be performed on rational numbers.
- Addition and subtraction of rational numbers can be performed after a common denominator has been identified, and the fractions have been converted to forms having the common denominator.
- Multiplication and division of rational numbers can be performed without regard to common denominators.
- Between any two rational numbers, there is always another rational number. This is the density property of the rational numbers.

Formulas

- $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$
- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$
- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \frac{a}{b} \div \frac{c}{d} = \frac{a}{d} \times \frac{d}{c} = \frac{a \times d}{b \times c}$

Videos

- Introduction to Fractions
- Equivalent Fractions
- Reducing Fractions to Lowest Terms
- Using Desmos to Reduce a Fraction
- Adding and Subtracting Fractions with Different Denominators

- Converting an Improper Fraction to a Mixed Number Using Desmos
- Multiplying Fractions
- Dividing Fractions
- Order of Operations Using Fractions
- Finding a Fraction of a Total
- Converting Units

3.5 Irrational Numbers

Figure 28: The Pythagoreans were a philosophical sect of ancient Greece, often associated with mathematics.

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are irrational.
2. Simplify irrational numbers and express in lowest terms.
3. Add and subtract irrational numbers.
4. Multiply and divide irrational numbers.
5. Rationalize fractions with irrational denominators.

The Pythagoreans were a philosophical sect in ancient Greece. Their philosophy included reincarnation and purifying the mind through the study and contemplation of mathematics and science. One of their principles was the cosmos is ruled by order, specifically mathematics and music. They even held mystic beliefs about specific numbers and figures. For example, the number 1 was associated with the mind and essence. Four represented justice, as it is the first product of two even numbers. Most famously, though, is the association with the Pythagorean Theorem, which states that in a right triangle, the sum of the squares of the shorter sides of the triangle (the legs) equals the square of the longer side (the hypotenuse). Even the ancient Egyptians used this relationship, as triangles with side measures 3, 4, and 5 were often used in surveying following the flooding of the Nile.

There is a story of a Pythagorean, Hippasus, discovering that not all numbers could be expressed as fractions. In other words, not all numbers were rational numbers. The story ends with Hippasus, who shared this, or in some versions discovered it, put to death by drowning for sharing this fact, that not all quantities could be expressed as the ratio of two natural numbers.

As colorful as that story may be, it is most likely false, as there are no contemporary sources to corroborate it. But it does seem to mark the discovery that not all quantities or measures were fractions of numbers. And so, **irrational numbers** were discovered.

VIDEO

The Philosophy of the Pythagoreans

Defining and Identifying Numbers That Are Irrational

We defined rational numbers in the last section as numbers that could be expressed as a fraction of two integers. **Irrational numbers** are numbers that cannot be expressed as a fraction of two integers. Recall that rational numbers could be identified as those whose decimal representations either terminated (ended) or had a repeating pattern at some point. So irrational numbers must be those whose decimal representations do not terminate or become a repeating pattern.

One collection of irrational numbers is **square roots** of numbers that aren't **perfect squares**. x is the square root of the number a, denoted $\sqrt{a}$, if $x^2 = a$. The number a is the perfect square of the integer n if $a = n^2$. The rational number $\frac{a}{b}$ is a perfect square if both a and b are perfect squares.

One method of determining if an integer is a perfect square is to examine its prime factorization. If, in that factorization, all the prime factors are raised to even powers, the integer is a perfect square. Another method is to attempt to factor the integer into an integer squared. It is possible that you recognize the number as a perfect square (such as 4 or 9). Or, if you have a calculator at hand, use the calculator to determine if the square root of the integer is an integer.

Example 1 **Identifying Perfect Squares**

Determine which of the following are perfect squares.

1. 45
2. 81
3. $\frac{9}{28}$
4. $\frac{144}{400}$

Solution

1. The prime factorization of 45 is $45 = 3^2 \times 5$. Since the 5 is not raised to an even power, 45 is not a perfect square.
2. The prime factorization of 81 is 3^4. All the prime factors are raised to even powers, so 81 is a perfect square.

3. We must determine if both the numerator and denominator of $\frac{9}{28}$ are perfect squares for the rational number to be a perfect square. The numerator is 9, and as mentioned above, 9 is a perfect square (it is 3 squared). Now we check the prime factorization of the denominator, 28, which is $28 = 2^2 \times 7$. Since 7 is not raised to an even power, 28 is not a perfect square. Since the denominator is not a perfect square, $\frac{9}{28}$ is not a perfect square.
4. We must determine if both the numerator and denominator of $\frac{144}{400}$ are perfect squares for the rational number to be a perfect square. The numerator is 144. The prime factorization of 144 is $144 = 2^4 \times 3^2$. Since all the prime factors of 144 are raised to even powers, 144 is a perfect square. Now we check the prime factorization of the denominator, 400, which is $400 = 2^4 \times 5^2$. Since all the prime factors of 400 are raised to even powers, 400 is a perfect square. Since the numerator and denominator of $\frac{144}{400}$ are perfect squares, $\frac{144}{400}$ is a perfect square.

TECH CHECK

Using Desmos to Determine if a Number Is a Perfect Square

Desmos may be used to determine if a number is a perfect square by using its square root function. When Desmos is opened, there is a tab in the lower left-hand corner of the Desmos screen. This tab opens the Desmos keypad, shown.

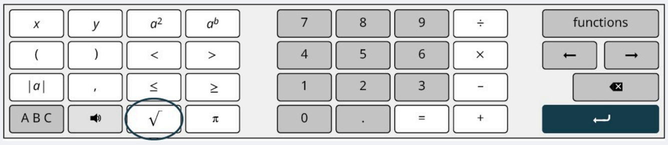


Figure 29: Desmos keyboard with square root key circles

There you find the key for the square root, which is circled in . To find the square root of a number, click the square root key, which begins a calculation, and then enter the value for which you want a square root. If the result is an integer, then the number is a perfect square.

VIDEO

Using Desmos to Find the Square Root of a Number

Another collection of irrational numbers is based on the special number, **pi**, denoted by the Greek letter π, which is the ratio of the circumference of the diameter of the circle .

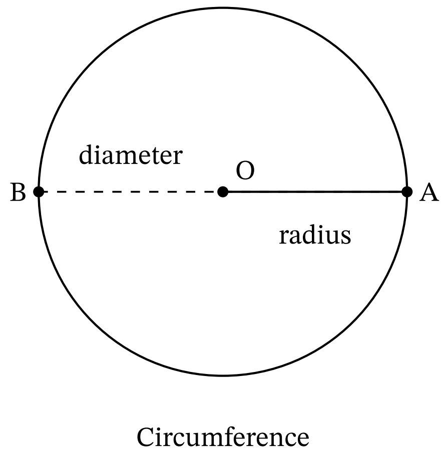


Figure 30: Circle with radius, diameter, and circumference labeled

Any multiple or power of π is an irrational number.

Any number expressed as a rational number times an irrational number is an irrational number also. When an irrational number takes that form, we call the rational number the **rational part**, and the irrational number the **irrational part**. It should be noted that a rational number plus, minus, multiplied by, or divided by any irrational number is an irrational number.

Example 2 **Identifying Irrational Numbers**

Identify which of the following numbers are irrational.

1. $\sqrt{35}$
2. $0.\overline{15}$
3. $\sqrt{121}$
4. 4π

Solution

1. 35 can be factored as 5×7, showing that 35 is not the square of an integer or a rational number. This mean its square root is an irrational number.
2. Since $0.\overline{15}$ is a decimal with a repeating pattern, it is rational, so it is not an irrational number.
3. $121 = 11^2$. Since 121 is the square of an integer, its square root is a rational number.
4. Since 4π is a multiple of pi, it is irrational. In this case, the rational part of the number is 4, while the irrational part is π.

WHO KNEW?

Euler-Mascheroni Constant

> Determining if a number is rational or irrational is not trivial. There are numbers that defied such classification for quite a long time. One such is the Euler-Mascheroni constant. The Euler-Mascheroni constant is used in mathematics, and is primarily associated with the natural logarithm, which is a mathematical function. The constant has been around since around 1790. However, it was unknown if this constant was rational or irrational until 2013, at which point it was proven to be irrational.

Simplifying Square Roots and Expressing Them in Lowest Terms

To **simplify a square root** means that we rewrite the square root as a rational number times the square root of a number that has no perfect square factors. The act of changing a square root into such a form is simplifying the square root.

The number inside the square root symbol is referred to as the **radicand**. So in the expression $\sqrt{a}$ the number a is referred to as the radicand.

Before discussing how to simplify a square root, we need to introduce a rule about square roots. The square root of a product of numbers equals the product of the square roots of those number. Written symbolically, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

> **FORMULA**
> For any two numbers a and b, $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

Using this formula, we can factor an integer inside a square root into a perfect square times another integer. Then the square root can be applied to the perfect square, leaving an integer times the square root of another integer. If the number remaining under the square root has no perfect square factors, then we've simplified the irrational number into lowest terms. To simplify the irrational number into lowest terms when n is an integer:

Step 1: Determine the largest perfect square factor of n, which we denote a^2.

Step 2: Factor n into $a^2 \times b$.

Step 3: Apply $\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b}$.

Step 4: Write $\sqrt{n}$ in its simplified form, $a\sqrt{b}$.

When a square root has been simplified in this manner, a is referred to as the rational part of the number, and $\sqrt{b}$ is referred to as the irrational part.

Example 3 **Simplifying a Square Root**

Simplify the irrational number $\sqrt{180}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 180. We can do this by writing out the factor pairs of 180:

$1 \times 180 \quad 2 \times 90 \quad 3 \times 60 \quad 4 \times 45 \quad 5 \times 36 \quad 6 \times 30 \quad 9 \times 20 \quad 10 \times 18 \quad 12 \times 15$

Looking at the list of factors, the perfect squares are 4, 9, and 36. The largest is 36, so we factor the into $36 \times 5 = 6^2 \times 5$. In the formula, $a = 6$ and $b = 5$. Apply $\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b}$.

$$\sqrt{6^2 \times 5} = \sqrt{6^2} \times \sqrt{5}$$

The simplified form of $\sqrt{180}$ is $6\sqrt{5}$. In this example, the 6 is the rational part, and the $\sqrt{5}$ is the irrational part.

VIDEO

Simplifying Square Roots

Example 4 **Simplifying a Square Root**

Simplify the irrational number $\sqrt{330}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 330. We can do this by writing out the factor pairs of 330:

$$1 \times 330 \quad 2 \times 165 \quad 3 \times 110 \quad 5 \times 66 \quad 6 \times 55 \quad 10 \times 33 \quad 11 \times 30 \quad 15 \times 22$$

Looking at the list of factors, there are no perfect squares other than 1, which means $\sqrt{330}$ is already expressed in lowest terms. In this case, 1 is the rational part, and $\sqrt{330}$ is the irrational part. **Though we could write this as $1\sqrt{330}$, but the product of 1 and any other number is just the number.**

Example 5 **Simplifying a Square Root**

Simplify the irrational number $\sqrt{2{,}548}$ and express in lowest terms. Identify the rational and irrational parts.

Solution

Begin by finding the largest perfect square that is a factor of 2,548. We can do this by writing out the factor pairs of 2,548:

$$1 \times 2548 \quad 2 \times 1274 \quad 4 \times 637 \quad 7 \times 364 \quad 13 \times 196 \quad 14 \times 182 \quad 26 \times 98 \quad 28 \times 91 \quad 49 \times 52$$

Looking at the list of factors, the perfect squares are 4, 49, and 196. The largest is 196, so we factor the 2,548 into $196 \times 13 = 14^2 \times 13$. In the formula, $a = 14$ and $b = 5$. Apply $\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b}$.

$$\sqrt{14^2 \times 13} = \sqrt{14^2} \times \sqrt{13}$$

The simplified form of $\sqrt{2{,}548}$ is $14\sqrt{13}$. In this example, 14 is the rational part, and $\sqrt{13}$ is the irrational part.

VIDEO

Simplifying Square Roots

Adding and Subtracting Irrational Numbers

Just like any other number we've worked with, irrational numbers can be added or subtracted. When working with a calculator, enter the operation and a decimal representation will be given. However, there are times when two irrational numbers may be added or subtracted without the calculator. This can happen only when the irrational parts of the irrational numbers are the same.

To add or subtract two irrational numbers that have the same irrational part, add or subtract the rational parts of the numbers, and then multiply that by the common irrational part.

FORMULA

Let our first irrational number be $a \times x$, where a is the rational and x the irrational parts.

Let the other irrational number be $b \times x$, where b is the rational and x the irrational parts.

Then $a \times x \pm b \times x = (a \pm b) \times x$.

Example 6 **Subtracting Irrational Numbers with Similar Irrational Parts**

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$$3\sqrt{7} - 8\sqrt{7}$$

Solution

Since these two irrational numbers have the same irrational part, $\sqrt{7}$, we can subtract without using a calculator. The rational part of the first number is 3. The rational part of the second number is 8. Using the formula yields $3\sqrt{7} - 8\sqrt{7} = (3 - 8) \times \sqrt{7} = -5\sqrt{7}$.

Example 7 **Adding Irrational Numbers with Similar Irrational Parts**

If possible, add the following irrational numbers without using a calculator. If this is not possible, state why.

$$35\pi + 17\pi$$

Solution

Since these two irrational numbers have the same irrational part, π, the addition can be performed without using a calculator. The rational part of the first number is 35. The rational part of the second number is 17. Using the formula yields $35\pi + 17\pi = (35 + 17) \times \pi = 52\pi$.

Example 8 **Subtracting Irrational Numbers with Different Irrational Parts**

If possible, subtract the following irrational numbers without using a calculator. If this is not possible, state why.

$$19\sqrt{3} - 5.6\sqrt{7}$$

Solution

The two numbers being subtracted do not have the same irrational part, so the operation cannot be performed.

Multiplying and Dividing Irrational Numbers

Just like any other number that we've worked with, irrational numbers can be multiplied or divided. When working with a calculator, enter the operation and a decimal representation will be given. Sometimes, though, you may want to retain the form of the irrational number as a rational part times an irrational part. The process is similar to adding and subtracting irrational numbers when they are in this form. We do not need the irrational parts to match. Even though they need not match, they do need to be similar, such as both irrational parts are square roots, or both irrational parts are multiples of pi. Also, if the irrational parts are square roots, we may need to reduce the resulting square root to lowest terms.

When multiplying two square roots, use the following formula. It is the same formula presented during the discussion of simplifying square roots.

FORMULA

For any two positive numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.

When dividing two square roots, use the following formula.

FORMULA

For any two positive numbers a and b, with b not equal to 0, $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.

To multiply or divide irrational numbers with similar irrational parts, do the following:

Step 1: Multiply or divide the rational parts.

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This becomes the rational part of the answer.

Step 3: Multiply or divide the irrational parts.

Step 4: If necessary, reduce the result from Step 3 to lowest terms. This becomes the irrational part of the answer.

Step 5: The result is the product of the rational and irrational parts.

Example 9 **Dividing Irrational Numbers with Similar Irrational Parts**

Perform the following operations without a calculator. Simplify if possible.

1. $3\sqrt{15} \div \left(8\sqrt{3}\right)$

2. $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$.

Solution

1. In this division problem, $3\sqrt{15} \div \left(8\sqrt{3}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above. **Step 1:** Divide the rational parts. $3 \div 8 = \frac{3}{8}$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. The 3 and 8 have no common factors, so $\frac{3}{8}$ is already in lowest terms.

Step 3: Divide the irrational parts. $\sqrt{15} \div \sqrt{3} = \frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}}$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{5}$.

Step 5: The result is the product of the rational and irrational parts, which is $\frac{3}{8}\sqrt{5}$.

1. In this division problem, $14.7\sqrt{135} \div \left(3\sqrt{5}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots, so follow the steps given above. **Step 1:** Divide the rational parts. $14.7 \div 3 = 4.9$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal so will not be reduced.

Step 3: Divide the irrational parts. $\sqrt{135} \div \sqrt{5} = \frac{\sqrt{135}}{\sqrt{5}} = \sqrt{\frac{135}{5}}$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand can be reduced, which yields $\sqrt{\frac{135}{5}} = \sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$.

Step 5: The result is the product of the rational and irrational parts, which is $4.9 \times 3\sqrt{3} = 14.7\sqrt{3}$.

Example 10 **Multiplying Irrational Numbers with Similar Irrational Parts**

Perform the following operations without a calculator. Simplify if possible.

1. $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$
2. $13\pi \times 8\pi$

Solution

1. In this multiplication problem, $\left(19\sqrt{3}\right) \times \left(5.6\sqrt{12}\right)$, notice that the irrational parts of these numbers are similar. They are both square roots. Follow the process above. **Step 1:** Multiply the rational parts. $19 \times 5.6 = 106.4$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. This rational number is expressed as a decimal and will not be reduced.

Step 3: Multiply the irrational parts. $\sqrt{3} \times \sqrt{12} = \sqrt{3 \times 12} = \sqrt{36}$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. The radicand is 36, which is the square of 6. The irrational part reduces to $\sqrt{36} = 6$.

Step 5: The result is the product of the rational and irrational parts, which is $106.4 \times 6 = 638.4$.

Notice that sometimes multiplying or dividing irrational numbers can result in a rational number.

1. In this multiplication problem, $13\pi \times 8\pi$, notice that the irrational parts of these numbers are the same, π. Follow the process above.

Step 1: Multiply the rational parts. $13 \times 8 = 104$

Step 2: If necessary, reduce the result of Step 1 to lowest terms. That result is an integer.

Step 3: Multiply the irrational parts. $\pi \times \pi = \pi^2$

Step 4: If necessary, reduce the result from Step 3 to lowest terms. This cannot be reduced.

Step 5: The result is the product of the rational and irrational parts, which is $104\pi^2$.

Rationalizing Fractions with Irrational Denominators

Fractions often represent that some amount is being equally divided into some number of parts. But to conceptualize a fraction in that manner, the denominator needs to be an integer. An irrational number in the denominator interferes with that interpretation of a fraction. Fractions that have denominators that are just the square root of an integer can be altered into fractions with integer denominators using a process called **rationalizing the denominator**. The process relies on the following property of square roots: $\sqrt{a} \times \sqrt{a} = a$ and the following property of fractions: $\frac{a}{b} = \frac{ac}{bc}$ for any non-zero number c.

Using these two properties, when a fraction has a square root in the denominator, we can eliminate that square root. Multiply the numerator and denominator by that square root from the denominator, $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}}$. Then apply $\sqrt{a} \times \sqrt{a} = a$ to the denominator, yielding $\frac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \frac{a\sqrt{b}}{b}$. Notice that there is no longer a square root in the denominator, which allows for interpreting the fraction as dividing a whole into equal parts.

VIDEO

Rationalizing the Denominator

Example 11 **Rationalizing the Denominator**

Rationalize the denominator of the following:

1. $\frac{5}{\sqrt{7}}$
2. $\frac{3\sqrt{6}}{2\sqrt{10}}$

Solution

1. The square root in the denominator is $\sqrt{7}$. In order to rationalize the denominator of $\frac{5}{\sqrt{7}}$, we need to multiply the numerator and denominator by $\sqrt{7}$ and simplify.
   $\frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{\sqrt{7}\times\sqrt{7}} = \frac{5\sqrt{7}}{7}$
   The square root is in simplified form, so the final answer is $\frac{5\sqrt{7}}{7}$.
2. The square root in the denominator is $\sqrt{10}$.
   Step 1: In order to rationalize the denominator of $\frac{3\sqrt{6}}{2\sqrt{10}}$, we need to multiply the numerator and denominator by $\sqrt{10}$ and simplify.
   $\frac{3\sqrt{6}}{2\sqrt{10}} = \frac{3\sqrt{6}\times\sqrt{10}}{2\sqrt{10}\times\sqrt{10}} = \frac{3\sqrt{60}}{2\times10} = \frac{3\sqrt{60}}{20}$ **Step 2:** The 60 under the square root can be factored into the following factor pairs:
   1×60 2×30 3×20 4×15 5×12 6×10 **Step 3:** The largest square factor of 60 is 4, so we simplify the $\sqrt{60}$ in the numerator into $2\sqrt{15}$. We also cancel any common factors.
   $\frac{3\sqrt{60}}{20} = \frac{3\times2\sqrt{15}}{20} = \frac{6\sqrt{15}}{20} = \frac{3\sqrt{15}}{10}$ This is completely simplified.

There are occasions when the denominator is irrational but is the sum of two numbers where one or both involve square roots. For instance, $\frac{5}{4+\sqrt{3}}$. The process used earlier required that the denominator was the square root of a number and would not work here. However, this type of denominator can be rationalized. In order to rationalize such a denominator, we will multiply the numerator and denominator of the fraction by the **conjugate** of the denominator. The conjugate of $a + b$ is $a - b$. We say that $a + b$ and $a - b$ are **conjugate numbers**.

So, the conjugate of $-3 + \sqrt{10}$ is just $-3 - \sqrt{10}$. But why is this of interest? The reason is because it leads to the **difference of squares** formula, which is used to factor the difference of two squares. Or, for our purposes, in reverse it allows us to eliminate a square root.

FORMULA

For any two numbers, a and b, $a^2 - b^2 = (a - b)(a + b)$.

Looking at that formula, you should see that the two factors on the right-hand side of the equals sign are conjugates of one another. So, for our purposes, we're interested in $(a - b)(a + b) = a^2 - b^2$. This tells us that when we multiply $a + b$ by its conjugate, we get a squared minus b squared, or $a^2 - b^2$. But how is this useful? Let's return to the fraction above, $\frac{5}{4+\sqrt{3}}$. The denominator is $4 + \sqrt{3}$. Its conjugate is $4 - \sqrt{3}$. According to the formula, and letting $a = 4$ and $b = \sqrt{3}$, we see that $\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right) = 4^2 - \left(\sqrt{3}\right)^2$. But $\left(\sqrt{3}\right)^2$ is just 3. That means the product is $16 - 3$ or 13. This no longer has a square root. We use this to rationalize the denominator.

We will also need the **distributive property** of numbers.

FORMULA
For any three numbers a, b, and c, $a \times (b \pm c) = a \times b \pm a \times c$. This is called the distributive property.

Example 12 **Rationalizing the Denominator Using Conjugates**
Rationalize the denominator of $\frac{4}{6+\sqrt{10}}$.
Solution
Step 1: We recognize that the denominator is the sum of two numbers where one or both involve square roots. This means the conjugate can be used to remove the square root from the denominator.

Step 2: To do so, we multiply the numerator and the denominator each by the conjugate of the denominator. Since the denominator is $6+\sqrt{10}$, the conjugate we will use is $6-\sqrt{10}$.

Step 3: The conjugate is multiplied by the numerator and the denominator.

$$\frac{4}{6+\sqrt{10}} \times \frac{6-\sqrt{10}}{6-\sqrt{10}}$$

Step 4: Remembering how a number times its conjugate works, this becomes

$$\frac{4}{6+\sqrt{10}} \times \frac{6-\sqrt{10}}{6-\sqrt{10}} = \frac{4 \times \left(6-\sqrt{10}\right)}{6^2-\left(\sqrt{10}\right)^2}$$

.

Step 5: In the numerator, we apply the distributive property. Using it yields $\frac{4\times\left(6-\sqrt{10}\right)}{6^2-\left(\sqrt{10}\right)^2} = \frac{24-4\sqrt{10}}{36-10} = \frac{24-4\sqrt{10}}{26}$.

Step 6: Notice that the denominator no longer contains a square root. It has been rationalized. If desired, this can then be written as a rational number minus an irrational number, by recalling that $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$.
Applying that to the answer, we have $\frac{4}{6+\sqrt{10}} = \frac{24-4\sqrt{10}}{26} = \frac{24}{26} - \frac{4\sqrt{10}}{26}$.

Step 7: With a bit of cancellation, this reduces to $\frac{4}{6+\sqrt{10}} = \frac{24}{26} - \frac{4\sqrt{10}}{26} = \frac{12}{13} - \frac{2\sqrt{10}}{13}$.

VIDEO
Rationalizing the Denominator

Key Terms

- conjugate numbers
- difference of squares

- irrational numbers
- lowest terms
- rationalize the denominator

Key Concepts

- Irrational numbers are numbers that cannot be written as an integer divided by another integer. One example is pi, denoted π. Another collection of irrational numbers are natural numbers that are not perfect squares.
- Some irrational numbers can be written as a rational part multiplied by an irrational part. If two irrational numbers have the same irrational parts, they can be added or subtracted.
- When irrational numbers are similar, on can multiply and divide the numbers without a calculator.
- Since $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, and $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, products and quotients of square roots can be determined.
- Because $\sqrt{a^2} = a$ and $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, it is possible to simplify square root expressions so the radicand contains no perfect square factors.
- When a fraction has an irrational number as its denominator, it is possible to convert the denominator into a rational number using its conjugate. Doing so involves multiplying the numerator and denominator by the conjugate of the denominator, and then applying the difference of squares formula.
- With a single square root term
- Using conjugate numbers for two term denominators

Formulas

- $$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$
- $$a \times x \pm b \times x = (a \pm b) \times x$$
- $$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$
- $$a^2 - b^2 = (a - b)(a + b)$$

Videos

- The Philosophy of the Pythagoreans
- Using Desmos to Find the Square Root of a Number
- Simplifying Square Roots
- Rationalizing the Denominator

3.6 Real Numbers

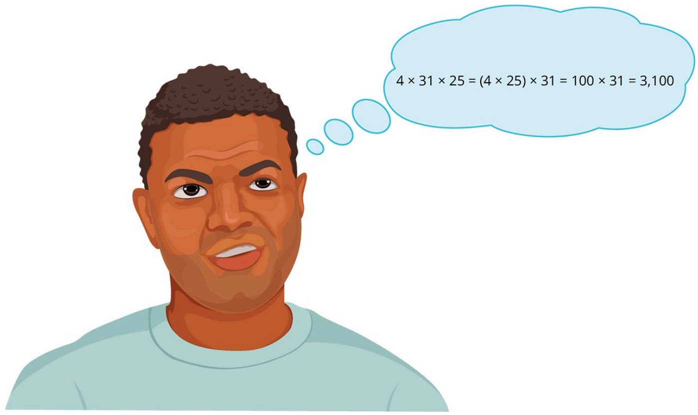


Figure 31: Quick mental math involves using the known properties of real numbers.

Learning Objectives

After completing this section, you should be able to:

1. Define and identify numbers that are real numbers.
2. Identify subsets of the real numbers.
3. Recognize properties of real numbers.

Have you ever been impressed by the speed at which someone can do math in their head? Most of us at one time or another have witnessed a person speed through mental math, an impressive feat that often bests calculators. One such person is Neelkantha Bhanu Prakash. As of September 20, 2020, he is considered the world's fastest human calculator. He currently holds four world records. How does someone do that, though? Have they memorized lots of arithmetic facts? Are they simply brilliant?

The answer isn't simple so much as it is about knowledge. Real numbers behave in some very regular ways, following rules that can be learned. In this section, those rules are explored.

Watch the video of Arthur Benjamin's TED Talk to learn about another mathematician with remarkable mental abilities.

VIDEO

Arthur Benjamin TED Talk, Faster Than a Calculator

Defining and Identifying Real Numbers

Real numbers are the rational and irrational numbers combined. The real numbers represent the collection of all physical distances that exist, along with 0 and the negatives of those physical distances. For example, if you take a measure of three units, and divide that distance into eight (8) equal lengths, the distance you have formed is $\frac{3}{8}$ units. Also, if you draw a **right triangle** (a triangle with one angle equal to 90 degrees) with one side length of 1, and the other side length of 3, the long side of the triangle will have length $\sqrt{10}$ units, as shown.

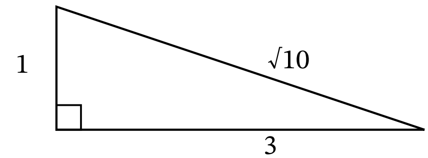


Figure 32: Right triangle

Of course, if we name something the real numbers, there must be numbers that aren't real. Otherwise, they'd just be called the numbers. One such not real number, one that cannot be a length, is $\sqrt{-1}$. It is part of a collection of numbers called the complex numbers, it is denoted with the letter i. As an extension, the square root of any negative number is not a real number, but instead a complex number.

To determine if a number is real, check to see if there are any negatives under a square root or any i's. If there are any present, the number is not real.

Example 1 **Identifying Real Numbers**

Determine if each of the following are real numbers:

1. $\frac{4\sqrt{3}}{7}$
2. 13.3381
3. $17\sqrt{-8}$

Solution

1. $\frac{4\sqrt{3}}{7}$ is a real number, as there are no negatives under the square roots, nor is there any factor of i.
2. 13.3381 is a rational number, and so it is a real number.
3. $17\sqrt{-8}$ is not a real number, as there is a negative number under the square root.

Identifying Subsets of Real Numbers

The real numbers were built out of pieces, including integers, rational numbers, and irrational numbers. As such, the real numbers have named subsets, as shown in the table below.

Set Name	Set Symbol	Set Description
Natural Numbers	$\mathbb{N}$	The counting numbers
Whole Numbers		The counting numbers and 0
Integers	$\mathbb{Z}$	The natural numbers, their negatives, and 0
Rational Numbers	$\mathbb{Q}$	Fractions of integers
Irrational Numbers	$\mathbb{P}$	Numbers that cannot be written as a fraction of integers
Real Numbers	$\mathbb{R}$	The union of the rational and irrational numbers, all possible physical lengths, and their negatives

When we categorize numbers using these sets, we use the smallest set that they belong to. For instance, -7 is an integer, and a rational number, and a real number. The smallest set to which -7 belongs is integer, so we'd say it belongs to the integers.

We can also represent the relationships between the different sets of real numbers using set notation. All natural numbers are integers, but there are integers that are not natural numbers, so $\mathbb{N} \subset \mathbb{Z}$. Similarly, every integer is a rational number, but there are rational numbers that are not integers, so $\mathbb{Z} \subset \mathbb{Q}$. The same is true of the rational numbers and the real numbers, so $\mathbb{Q} \subset \mathbb{R}$.

There is no agreed-upon symbol for the irrational numbers. If we represent the irrationals as the set A, we should note that the following are true: $\mathbb{Q} \cup A = \mathbb{R}$ and $\mathbb{Q} \cap A = \varnothing$. Recall that this means the irrationals are the complement of the rational numbers in the universal set of real numbers.

Example 2 **Categorizing Numbers**

Identify all subsets of the real numbers to which the following real numbers belong:

1. 14
2. -14.223
3. $\sqrt{17}$

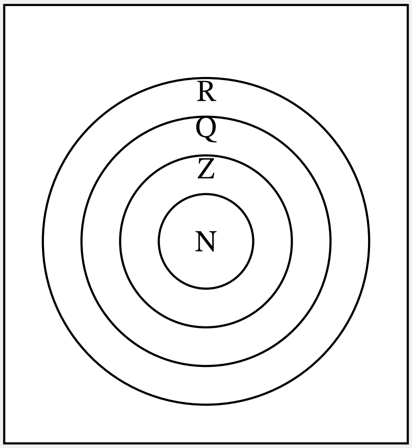
Solution

1. 14 is a natural number, integer, and rational number.
2. -14.223 is a rational number.
3. $\sqrt{17}$ is an irrational number.

Example 3 **Categorizing Numbers within a Venn Diagram**

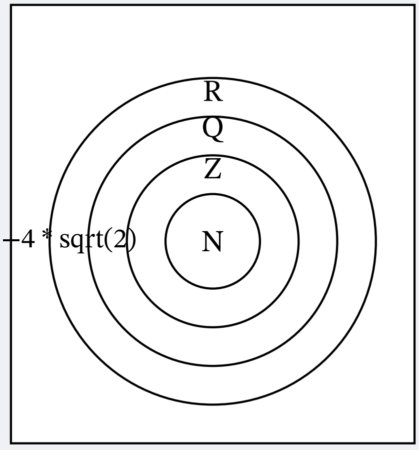
Place the following numbers correctly in the Venn diagram .

$$-4\sqrt{2} \quad -10 \quad \frac{37}{150} \quad 41 \quad \frac{1}{20} \quad 4\pi$$

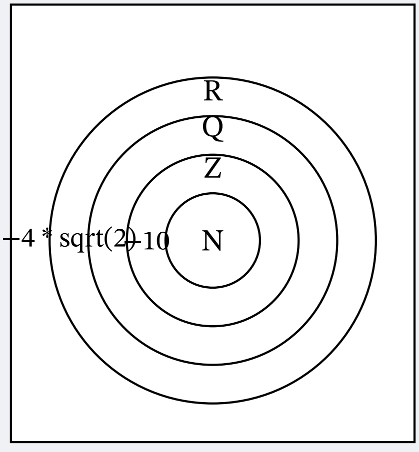


Solution

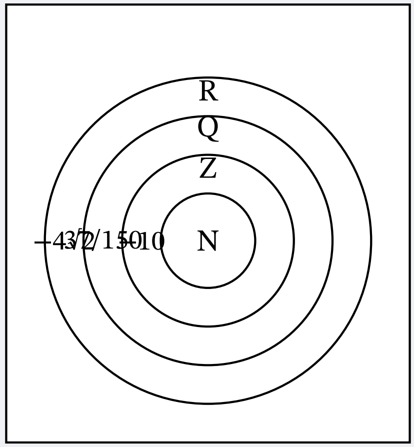
Since $-4\sqrt{2}$ is irrational, it belongs in the real numbers, but outside the rational numbers .



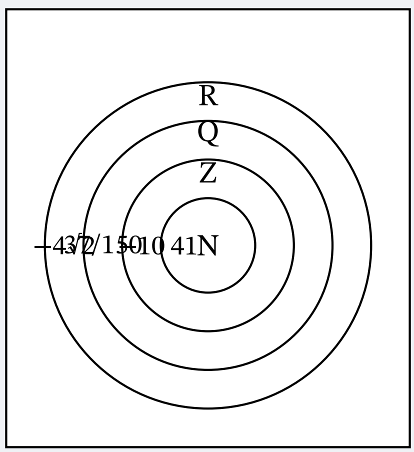
Since -10 is an integer, it belongs in the integers but outside the natural numbers .



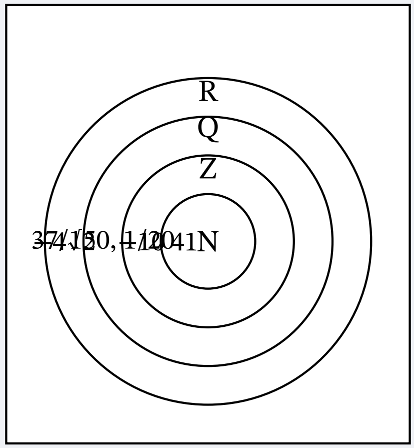
Since $\frac{37}{150}$ is a rational number, it belongs in the rational numbers but not in the integers .



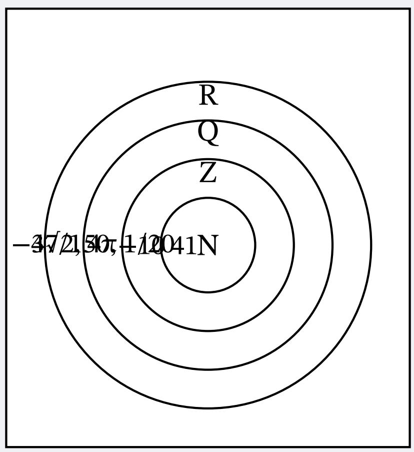
Since 41 is a natural number, it belongs in the natural numbers circle .



Since $\frac{1}{20}$ is a rational number, it belongs in the rational numbers but not in the integers .



Since 4π is irrational, it belongs in the real numbers, but outside the rational numbers .



VIDEO

Identifying Sets of Real Numbers

Recognizing Properties of Real Numbers

The real numbers behave in very regular ways. These behaviors are called the **properties of the real numbers**. Knowing these properties helps when evaluating formulas, working with equations, or performing algebra. Being familiar with these properties is helpful in all settings where numbers are used and manipulated. For example, when multiplying $4 \times 13 \times 25$, you could multiply the 4 and 25 first. If you know that product is 100, it makes the multiplication easier.

The table below is a partial list of properties of real numbers.

Property	Example	In Words
Distributive property $a \times (b + c) = a \times b + a \times c$	$5 \times (3 + 4) = 5 \times 3 + 5 \times 4$	Multiplication distributes across addition
Commutative property of addition $a + b = b + a$	$3 + 7 = 7 + 3$	Numbers can be added in any order
Commutative property of multiplication $a \times b = b \times a$	$10 \times 4 = 4 \times 10$	Numbers can be multiplied in any order
Associative property of addition $a + (b + c) = (a + b) + c$	$4 + (3 + 8) = (4 + 3) + 8$	Doesn't matter which pair of numbers is added first
Associative property of multiplication $a \times (b \times c) = (a \times b) \times c$	$2 \times (5 \times 7) = (2 \times 5) \times 7$	Doesn't matter which pair of numbers is multiplied first
Additive identity property $a + 0 = a$	$17 + 0 = 17$	Any number plus 0 is the number
Multiplicative identity property $a \times 1 = a$	$21 \times 1 = 21$	Any number times one is the number
Additive inverse property $a + (-a) = 0$	$14 + (-14) = 0$	Every number plus its negative is 0
Multiplicative inverse property $a \times \left(\frac{1}{a}\right) = 1$, provided $a \neq 0$	$3 \times \left(\frac{1}{3}\right) = 1$	Every non-zero number times its reciprocal is 1

The names of the properties are suggestive. The **commutative properties**, for example, suggest commuting, or moving. **Associative properties** suggest which items are associated with others, or if order matters in the computation. The **distributive property** addresses how a number is distributed across parentheses.

Example 4 **Identifying Properties of Real Numbers**

In each of the following, identify which property of the real numbers is being applied.

1. $4 + (8 + 13) = (4 + 8) + 13$
2. $34 \times \left(\frac{1}{34}\right) = 1$
3. $14 + 27 = 27 + 14$

Solution

1. Here, the pair of numbers that is added first is switched. This is the associative property of addition.
2. Here, a number is multiplied by its reciprocal, resulting in 1. This is the multiplicative inverse property.
3. Here, the order in which numbers are added is switched. This is the commutative property of addition.

Using these properties to perform arithmetic quickly relies on spotting easy numbers to work with. Look for numbers that add to a multiple of 10, or multiply to a multiple of 10 or 100.

Example 5 **Using Properties of Real Numbers in Calculations**

Use properties of the real numbers and mental math to calculate the following:

1. $2 \times 13 \times 50$
2. $13 + 84 + 27$
3. $9 \times 16 \times 11$

Solution

1. Notice that $2 \times 50 = 100$, so that becomes the multiplication to do first. Use the commutative property of multiplication to change the order of the numbers being multiplied.
$$2 \times 13 \times 50 = 2 \times 50 \times 13 = 100 \times 13 = 1,300$$
2. Notice that $13 + 27 = 40$, so that becomes the addition to do first. Use the commutative property of addition to change the order in which the numbers are added.
$$13 + 84 + 27 = 13 + 27 + 84 = 40 + 84 = 124$$
3. Notice that $9 \times 11 = 99$. Using that, the problem can be changed to 99×16. That, however, doesn't look easy at all. But $99 = (100 - 1)$. Using the distributive property, we rewrite and expand this as $99 \times 16 = (100 - 1) \times 16 = 100 \times 16 - 1 \times 16 = 1,600 - 16$. The last step is subtraction, so the final answer is 1,584. So, multiplying by 99 is the same as multiplying by 100, and then subtracting the other number once.

Videos

- Properties of the Real Numbers 1
- Properties of the Real Numbers 2
- Properties of the Real Numbers 3

Key Terms

- complex number
- imaginary number

- real number

Key Concepts

- Real numbers is the collection of all rational and irrational numbers. Conceptually, it is the collection of all values that can be represented on a number line, or, as a length along with sign.
- The subsets of the real numbers include the natural numbers, integers, rational numbers and irrational numbers. The natural numbers are a subset of the integers, which is a subset of the rational numbers. The rational and irrational numbers are disjoint sets.
- The real numbers, due to order of operation rules and that performing arithmetic operations on real number always results in a real number, have arithmetic properties that apply in all cases. There include the distributive property, the commutative property, and the associative property. Also, every real number has an additive inverse and, except for zero (0), have a multiplicative inverse.

Formulas

- $a \times (b + c) = a \times b + a \times c$
- $a + b = b + a$
- $a \times b = b \times a$
- $a + (b + c) = (a + b) + c$
- $a \times (b \times c) = (a \times b) \times c$
- $a + 0 = a$
- $a \times 1 = a$
- $a + (-a) = 0$
- $a \times \left(\frac{1}{a}\right) = 1$

Videos

- Arthur Benjamin TED talk, Faster than a Calculator
- Identifying Sets of Real Numbers
- Properties of the Real Numbers #1
- Properties of the Real Numbers #2
- Properties of the Real Numbers #3

3.7 Clock Arithmetic

Figure 40: If a

Learning Objectives

After completing this section, you should be able to:

1. Add, subtract, and multiply using clock arithmetic.
2. Apply clock arithmetic to calculate real-world applications.

   Online shopping requires you to enter your credit card number, which is then sent electronically to the vendor. Using an ATM involves sliding your bank card into a reader, which then reads, sends, and verifies your card. Swiping or tapping for a purchase in a brick–and-mortar store is how your card sends its information to the machine, which is then communicated to the store's computer and your credit card company. This information is read, recorded, and transferred many times. Each instance provides one more opportunity for error to creep into the process, a misrecorded digit, transposed digits, or missing digits. Fortunately, these card numbers have a built-in error checking system that relies on modular arithmetic, which is often referred to as clock arithmetic. In this section, we explore clock, or modular, arithmetic.

> **VIDEO**
> Determining the Day of the Week for Any Date in History

Adding, Subtracting, and Multiplying Using Clock Arithmetic

When we do arithmetic, numbers can become larger and larger. But when we work with time, specifically with clocks, the numbers cycle back on themselves. It will never be 49 o'clock. Once 12 o'clock is reached, we go back to 1 and repeat the numbers. If it's 11 AM and someone says, "See you in four hours," you know that 11 AM plus 4 hours is 3 PM, not 15 AM (ignoring military time for now). Math worked on the clock, where numbers restart after passing 12, is called **clock arithmetic**.

Clock arithmetic hinges on the number 12. Each cycle of 12 hours returns to the original time . Imagine going around the clock one full time. Twelve hours pass, but the time is the same. So, if it is 3:00, 14 hours later and two hours later both read the same on the clock, 5:00. Adding 14 hours and adding 2 hours are identical. As is adding 26 hours. And adding 38 hours.

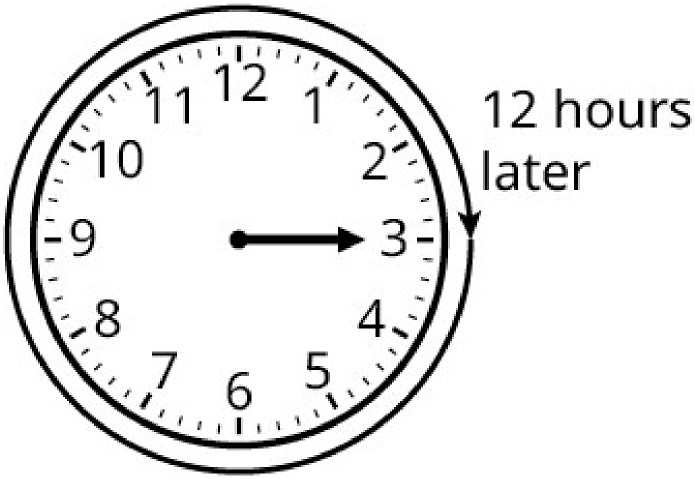


Figure 41: Clock showing 3:00 with arrow going around the clock one full time, or 12 hours

What do 2, 14, 26, and 38 have in common in relation to 12? When they are divided by 12, they each have a remainder of 2. That's the key. When you add a number of hours to a specific time on the clock, first divide the number of hours being added by 12 and determine the remainder. Add that remainder to the time on the clock to know what time it will be.

A good visualization is to wrap a number line around the clock, with the 0 at the starting time. Then each time 12 on the number line passes, the number line passes the starting spot on the clock. This is referred to as **modulo 12** arithmetic. Even though the process says to divide the number being added by 12, first perform the addition; the result will be the same if you add the numbers first, and then divide by 12 and determine the remainder.

In general terms, let n be a positive integer. Then n modulo 12, written (n mod 12), is the remainder when n is divided by 12. If that remainder is x, we would write $n = x \pmod{12}$.

CHECKPOINT

Caution: 12 mod 12 is 0. So, if a mod 12 problem ends at 0, that would be 12 on the clock.

Example 1 **Determining the Value of a Number modulo 12**

Find the value of the following numbers modulo 12:

1. 34
2. 539
3. 156

Solution

To determine the value of a number modulo 12, divide the number by 12 and record the remainder.

1. To find the value 34 modulo 12:

Step 1: Determine the remainder when 34 is divided by 12 using long division. The largest multiple of 12 that is less than or equal to 34 is 24, which is the product of 12 and 2.

$$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \end{array}$$

Step 2: Performing the subtraction yields 10.

$$\begin{array}{r} 2 \\ 12\overline{)34} \\ \underline{24} \\ 10 \end{array}$$

Since that subtraction resulted in a number less than 12, that is the remainder, 10. The value of 34 modulo 12 is 10, or $34 = 10 \pmod{12}$.

1. To find the value 539 modulo 12:
   Step 1: Determine the remainder when 539 is divided by 12 using long division. We first look to the first two digits of 539, 53. The largest multiple of 12 that is less than or equal to 53 is 48, which is the product of 12 and 4.

$$\begin{array}{rl} & 4 \\ 12) & \overline{539} \\ & \underline{48} \end{array}$$

Step 2: Performing the subtraction results in 5.

$$\begin{array}{rl} & 4 \\ 12) & \overline{539} \\ & \underline{48} \\ & 5 \end{array}$$

Step 3: Now, the 9 is brought down.

$$\begin{array}{rl} & 4 \\ 12) & \overline{539} \\ & \underline{48} \\ & 59 \end{array}$$

Step 4: The largest multiple of 12 that is less than or equal to 59 is once more 48 itself, which is 12×4.

$$\begin{array}{rl} & 44 \\ 12) & \overline{539} \\ & \underline{48} \\ & 59 \\ & \underline{48} \end{array}$$

Step 5: Finishing the process, the 48 is subtracted from the 59, yielding 11.

$$\begin{array}{r} 44 \\ 12\overline{)539} \\ \underline{48} \\ 59 \\ \underline{48} \\ 11 \end{array}$$

We've used all the digits of 539, and the last subtraction resulted in a number less than 12, so that number, 11, is the remainder. The value of 539 modulo 12 is 11, or, 539 = 11 (mod 12).

1. To find the value 156 modulo 12:
   Step 1: Determine the remainder when 156 is divided by 12 using long division. We first look to the first two digits of 156, 15. The largest multiple of 12 that is less than or equal to 15 is 12 itself, which is the product of 12 and 1.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \end{array}$$

Step 2: Performing the subtraction results in 3.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 3 \end{array}$$

Step 3: Now, the 6 is brought down.

$$\begin{array}{r} 1 \\ 12\overline{)156} \\ \underline{12} \\ 36 \end{array}$$

Step 4: The largest multiple of 12 that is less than or equal to 36 is 36 itself, which is 12×3.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \underline{12} \\ 36 \\ \underline{36} \end{array}$$

Step 5: Finishing the process, the 36 is subtracted from the 36, yielding 0.

$$\begin{array}{r} 13 \\ 12\overline{)156} \\ \underline{12} \\ 36 \\ \underline{36} \\ 0 \end{array}$$

We've used all the digits of 156, and the last subtraction resulted in a number less than 12, so that number, 0, is the remainder. The value of 156 modulo 12 is 0, or, $156 = 0$ (mod 12).

We should note here that, had we been speaking of time, the 0 would be interpreted as 12:00.

TECH CHECK

Using Desmos to Determine the Value of a Number module 12

Desmos may be used to determine the value of a number modulo 12. It is flexible enough to find the value of a number modulo of any other integer you want. To determine the value of n modulo 12, type **mod(n,12)** into Desmos. The result will be displayed immediately. This can be used to find 539 modulo 12, as shown in the .

Figure 42: Display of 539 modulo 12

Clock arithmetic is modulo 12 arithmetic but applied to time. As time is divided into 12 hours that repeat a cycle, we use modulo 12 for clock arithmetic.

VIDEO

Clock Arithmetic

Example 2 Adding with Clock Arithmetic

If it's 3:00, what time will it be in 89 hours?

Solution

To find that future time, we may determine the value of 89 (mod 12), either by long division or by using a calculator, such as Desmos. Then add the result to 3:00. Entering **mod(89,12)** in Desmos results in 5. Adding 5 hours, which was 89 (mod12), to 3:00 results in 8:00.

Subtracting time on the clock works in much the same way as addition. Find the value of the number of hours being subtracted modulo 12, then subtract that from the original time.

Example 3 Subtracting with Clock Arithmetic

If it is 4:00 now, what time was it 67 hours ago?

Solution

To find that past time, we may determine the value of 67 (mod 12), either by long division or by using a calculator, such as Desmos. Then subtract the result to 4:00. Entering **mod(67,12)** in Desmos results in 7. Subtracting 7 hours from 4:00 results in –3:00. We know, though, that time is not represented with negative times. This value, –3:00, indicates three hours before 12:00, which is 9:00. So, 67 hours before 4:00 was 9:00. We see this in the .

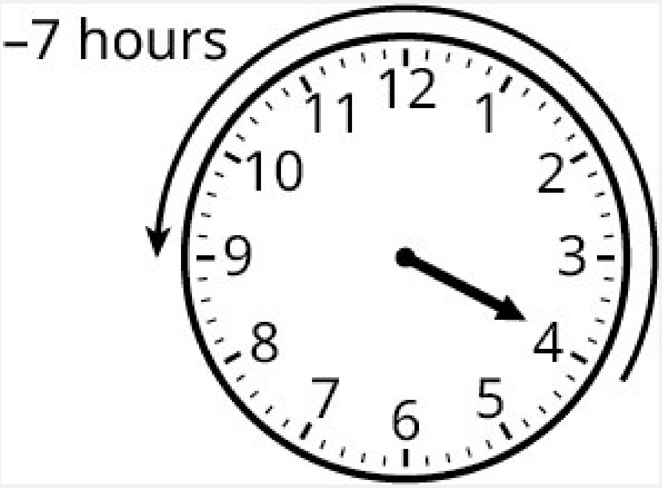


Figure 43: Clock showing 7 hours subtracted from 4:00

Recall that clock arithmetic was referred to as modulo 12 arithmetic. Multiplying in modulo 12 also relies on the remainder when dividing by 12. To multiply modulo 12 is just to multiply the two numbers, and then determine the remainder when divided by 12.

Example 4 **Multiplying modulo 12**

What is the product of 11 and 45 modulo 12?

Solution

We begin by multiplying 11 and 45, which is 495. Next, we find 495 modulo 12, either by dividing the result by 12 to determine the remainder, or by using a calculator. Entering **mod(495,12)** in Desmos yields 3. Had long division been used, the remainder would be 3. So $11 \times 45 = 3$ modulo 12.

Calculating Real-World Applications with Clock Arithmetic

Example 5 **Applying Clock Arithmetic**

Suppose it is 3:00, and you decide to check your email every 5 hours. What time will it be when you check your email the ninth time?

Solution

If you check your email every 5 hours nine times, that ninth check will occur 45 hours after 3:00, which is an addition of 45 hours to 3:00. So, we find 45 modulo 12, which is 9. Nine hours after 3:00 is 12:00. It will be 12:00 when you check your email the ninth time.

Clock arithmetic processes can be applied to days of the week. Every 7 days the day of the week repeats, much like every 12 hours the time on the clock repeats. The only difference will be that we work with remainders after dividing by 7. In technical terms, this is referred to as **modulo 7**. More generally, let n be a positive integer. Then n modulo 7, written n mod 7, is the remainder when n is divided by 7. If that value is x, we may write $n = x \pmod 7$.

Example 6 **Applying Clock Arithmetic to Days of the Week**

Your family has a cat, and no one wants to empty the litter box. However, it has to be done daily. The six of you agree to take turns, so everyone has to empty the litter box every 6 days. You empty the box on a Thursday. What day will you empty the box for the 10th time?

Solution

The first time you emptied the litter box was on a Thursday. So,the 10th time you empty the litter box will be 9 times later (you've already had your first turn, so 9 turns left!). This will happen 54 (9 times 6) days later. Finding the value of 54 modulo 7, using division to determine the remainder or using a calculator to find the value of 54 modulo 7 gives the answer 5. Five days after a Thursday is Tuesday.

Key Terms

- clock arithmetic
- modulo 7
- modulo 12

Key Concepts

- Clock arithmetic uses the idea that after 12 o'clock comes 1 o'clock. For clock arithmetic, this means that every time 12 is passed in an arithmetic process, the next number is 1, not 13.
- To determine the clock result of an arithmetic operation, divide the final result by 12 and keep the remainder. If the remainder is 0, then the time is 12 o'clock.
- Clock arithmetic is technically called modulo 12 arithmetic. To perform modulo 12 arithmetic, calculate the expression, then divide the result by 12. The modulo 12 result is the remainder.
- Days, in our system, pass in groups of seven. To calculate in day arithmetic, modulo 7 is used. To perform modulo 7 arithmetic, calculate the expression, then divide the result by 7. The modulo 7 result is the remainder.

Videos

- Determining the Day of the Week for Any Date in History
- Clock Arithmetic

3.8 Exponents

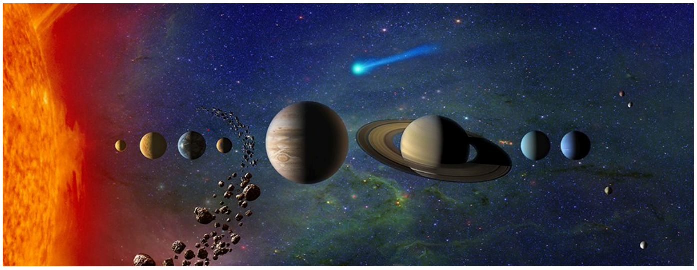


Figure 44: Astronomical distances are written using exponents.

Learning Objectives

After completing this section, you should be able to:

1. Apply the rules of exponents to simplifying expressions.

Sometimes, we look for shorthand when writing or expressing something that simply takes too long. The use of LOL and tl;dr. This shorthand only works if everyone reading the shorthand knows what it stands for. Using exponents is a similar instance. Writing out a long string of a number times itself over and over takes too much time, and eventually one would forget how many of the value has been written or read. For example, $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$. There has to be a shorter and more efficient way to write 8 times itself 1, 2, 3....hmmmm, 19 times.

And that's the role that exponents play in mathematics. They are shorthand for multiplying a number by itself a number of times. Without it, calculations would become a mess and we'd have to write a lot more.

Applying the Rules of Exponents to Simplify Expressions

Squaring a number is multiplying it by itself, and has that name because it is the area of a square with that side length. Cubing a number is finding the volume of a cube with that length of sides. That's why we refer to 5^2 as five squared, or 10^3 as ten cubed. Exponents represent that multiplication.

Let's remind ourselves of the terminology associated with exponents and what exponents represent. Suppose you want to multiply a number, let's label that number a, by itself some number of times. Let's label the number of times b. We denote that as a^b. We say a raised to the bth power. When we write or see 7^5, we call the 7 the **base** and we call 5 the **exponent**. What it represents is 7 multiplied by itself 5 times. This means exponents are used as a shorthand for repeated multiplications, where we write $7^5 = 7 \times 7 \times 7 \times 7 \times 7$. We would write 7^5 and say seven to the fifth power.

VIDEO

Exponential Notation

The definitions of base and exponent make it possible to understand the exponent rules.

Product Rule for Exponents

The first rule we examine is the product rule, $a^n a^m = a^{n+m}$. This rule means that when we multiply a base raised to a power times the same base to another power, the result is the base raised to the sum of the powers. To demonstrate, consider $9^3 \times 9^5$. If we apply the product rule to that we get $9^3 \times 9^5 = 9^{3+5} = 9^8$. This can be tested by looking at the multiplications that are represented. The 9^3 is 9 times itself 3 times, while 9^5 is 9 times itself 5 times. Substituting those into $9^3 \times 9^5$ we see $9^3 \times 9^5 = (9 \times 9 \times 9) \times (9 \times 9 \times 9 \times 9 \times 9) = 9^8$, which is what the formula told us would happen.

CHECKPOINT

Caution: The product rule only applies when the bases are the same. If the bases are different, we do not apply this rule.

FORMULA

If a number, a, raised to a power, n, is then multiplied by a raised to another power, m, the result is $a^n a^m = a^{n+m}$.

Example 1 **Using the Product Rule for Exponents**

If possible, use the product rule to simplify the following:

1. $21^9 \times 21^{15}$
2. $5^9 \times 8^4$

Solution

1. We can apply the product rule to simplify the expression because the bases are the same and we are multiplying.

$$21^9 \times 21^{15} = 21^{(9+15)} = 21^{24}$$

2. Since the bases are not the same (one is 5, the other 8), this cannot be simplified using the product rule for exponents.

These rules can be applied to unknowns too.

Example 2 **Using the Product Rule for Exponents of Unknowns**

Use the product rule to simplify $a^4 \times a^{10}$.

Solution

The bases are the same, and we are multiplying, so we apply the multiplication rule to simplify the expression.

$$a^4 \times a^{10} = a^{(4+10)} = a^{14}$$

Quotient Rule for Exponents

The next rule we examine is the quotient, or division, rule.

FORMULA

When a number, a, raised to a power, n, is divided by a raised to another power, m, then the result is $\frac{a^n}{a^m} = a^{(n-m)}$.

This rule means that when we divide a base raised to a power by the same base to another power, the result is the base raised to the difference of the powers. To demonstrate, consider $\frac{14^{13}}{14^6}$. If we apply the quotient rule to that, we get $\frac{14^{13}}{14^6} = 14^{13-6} = 14^7$. This can be tested by looking at the division that is represented. Remember, 14^{13} is 14 multiplied to itself 13 times, while 14^6 is 14 multiplied to itself 6 times. Substituting those into $\frac{14^{13}}{14^6}$ gives the following:

$$\frac{4^{13}}{4^6} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4 \times 4}$$

We see here that there are a LOT of fours to be divided out.

$$= \frac{\cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{\cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4} \times \cancel{4}} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{1} = 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$$

What remains is 4 to the 7th power, $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^7$.

All of the work above confirmed what the formula told us would be the result.

CHECKPOINT

Caution: The quotient rule only applies when the bases are the same. If the bases are different, we do not apply this rule.

Example 3 **Using the Quotient Rule for Exponents**

Use the quotient rule to simplify $\frac{5^{19}}{5^{11}}$.

Solution

We can apply the quotient rule to simplify the expression since the bases are the same and we are dividing.

$$\frac{5^{19}}{5^{11}} = 5^{(19-11)} = 5^8$$

VIDEO

Product and Quotient Rule for Exponents

A natural consequence of the quotient rule is what it means to raise a non-zero number to the zeroth power. Let’s look at the simplification when the exponents are equal.

$$\frac{3^6}{3^6} = 3^{(6-6)} = 3^0$$

We know that a number divided by itself is 1, so $\frac{3^6}{3^6} = 1$. From that is must be that $\frac{3^6}{3^6} = 3^0 = 1$. This provides the rule for a number raised to the power 0: $a \neq 0$.

FORMULA

If you have a non-zero number a, then $a^0 = 1$.

Distributive Rule for Exponents

The next rule we look to is a distributive rule for exponents.

FORMULA

If you have a product, $(a \times b)$, and raise it to an exponent, n, then $(a \times b)^n = a^n \times b^n$.

This means that when we have two numbers multiplied together, and that is raised to a power, it is the same as raising each of the numbers to the same power first, then multiplying. For example, $(3 \times 7)^4 = 3^4 \times 7^4$. This can be explained using the definition of exponents and multiplying all the factors.

$$(3 \times 7)^4 = (3 \times 7) \times (3 \times 7) \times (3 \times 7) \times (3 \times 7)$$

We may change the order in which numbers are multiplied. This is the commutative property of the real numbers. This can be written as $3 \times 3 \times 3 \times 3 \times 7 \times 7 \times 7 \times 7$. Using exponents, that shortens to $3^4 \times 7^4$.

This also works in the other direction, $a^n \times b^n = (a \times b)^n$. Read this way, if we have one base raised to an exponent, and another base raised to the same exponent, we can multiply the bases and raise that product to the shared exponent. For instance, $7^8 \times 11^8 = (7 \times 11)^8 = 77^8$.

CHECKPOINT

Caution: The exponent distributive rule, $a^n \times b^n = (a \times b)^n$, only works if the exponents are the same.

Example 4 **Using the Distributive Rule for Exponents**

Use the exponent distributive rule to expand $(6 \times 13)^7$.

Solution

Applying the distributive rule to the product, we get $(6 \times 13)^7 = 6^7 \times 13^7$.

Example 5 **Using the Distributive Rule for Exponents**

Use the exponent distributive rule to expand $(c \times d)^{10}$.

Solution

Applying the distributive rule to the product, we get $(c \times d)^{10} = c^{10} \times d^{10}$.

This distribution also works for quotients. A fraction raised to an exponent equals the numerator raised to the exponent divided by the denominator raised to the exponent. For example, $\left(\frac{3}{5}\right)^7 = \frac{3^7}{5^7}$. Demonstrating this is similar to the previous rule.

FORMULA

When you have a fraction, $\frac{a}{b}$, raised to an exponent, n, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Example 6 **Using the Distributive Rule for Exponents with Fractions**

Use the exponent distributive rule to expand the following:

1. $\left(\frac{4}{9}\right)^6$
2. $\left(\frac{3}{b}\right)^{11}$

Solution

1. Applying the distributive rule to the quotient, we get $\left(\frac{4}{9}\right)^6 = \frac{4^6}{9^6}$.
2. Applying the distributive rule to the quotient, we get $\left(\frac{3}{b}\right)^{11} = \frac{3^{11}}{b^{11}}$.

VIDEO

Fraction Raised to a Power

Power Rule

In the previous two sets of rules, we've seen exponents applied to products and quotients. Now we look to exponents applied to other exponents. For example, $\left(3^6\right)^4 = 3^{(6\times4)} = 3^{24}$. This can be explained by examining what the outer exponent does. We raise 3^6 to the fourth power, so we multiply 3^6 by itself 4 times, $\left(3^6\right)^4 = 3^6 \times 3^6 \times 3^6 \times 3^6$. Now if we apply the product rule for exponents, this becomes $3^{(6+6+6+6)} = 3^{24}$.

FORMULA

If you raise a non-zero base, say a, to an exponent n, and raise that to another exponent, m, you get the base raised to the product of the exponents, which is $(a^n)^m = a^{(n\times m)}$.

Example 7 **Raising an Exponent to an Exponent**

Expand the following:

1. $\left(6^7\right)^3$
2. $\left(b^{12}\right)^4$

Solution

1. Using the power rule of exponents, $\left(6^7\right)^3 = 6^{(7\times3)} = 6^{21}$.
2. Using the power rule of exponents, $\left(b^{12}\right)^4 = b^{(12\times4)} = b^{48}$.

Negative Exponent Rule

Up until now, we've only looked at positive exponents. The last exponent rule we look at is what negative exponents represent. Recall the quotient rule: $\frac{a^n}{a^m} = a^{(n+m)}$. What would happen if the exponent in the denominator was larger than that in the numerator? For example, $\frac{4^5}{4^7}$. If we apply the quotient rule, we obtain $\frac{4^5}{4^7} = 4^{5-7} = 4^{-2}$. We need to make sense of that negative exponent. To do so, we can expand the quotient and see what happens: $\frac{4^5}{4^7} = \frac{4\times4\times4\times4\times4}{4\times4\times4\times4\times4\times4\times4}$. When we divide out common factors, only two factors of 4 are left in the denominator, as we see here: $\frac{1}{4\times4}$. Using exponent notation, this is $\frac{1}{4^2}$. Since 4⁻2 and $\frac{1}{4^2}$ represent the same number, $\frac{4^5}{4^7}$, they are equal. This demonstrates how negative exponents are defined.

FORMULA

$a^{-n} = \frac{1}{a^n}$ provided that $a \neq 0$.

Similarly, $\frac{1}{a^{-n}} = a^n$.

Example 8 **Eliminating Negative Exponents**

Convert the following to expressions with no negative exponent:

1. $3^4 \times 5^{-8}$
2. $a^{-9} \times b^5$
3. $\frac{7}{c^{-2}}$

Solution

1. Using the negative exponent rule on the 5^{-8} and multiplying, $3^4 \times 5^{-8} = 3^4 \times \frac{1}{5^8} = \frac{3^4}{5^8}$.
2. Using the negative exponent rule on the a^{-9} and multiplying, $a^{-9} \times b^5 = \frac{1}{a^9} \times b^5 = \frac{b^5}{a^9}$.
3. Begin by rewriting the expression as $\frac{7}{c^{-2}} = \frac{7}{1} \times \frac{1}{c^{-2}}$. Apply the negative exponent rule to $\frac{1}{c^{-2}}$ in the expression, which becomes $\frac{7}{1} \times \frac{1}{c^{-2}} = 7 \times c^2$, which has no negative exponents.

Example 9 **Eliminating Denominators by Using Negative Exponents**

Use negative exponents to rewrite the following expressions with no denominator:

1. $\frac{7^3}{13^9}$
2. $\frac{c^4}{d^8}$

Solution

1. Rewrite the expression $\frac{7^3}{13^9}$ as $\frac{7^3}{1} \times \frac{1}{13^9}$. Then use the definition of negative exponents to rewrite the $\frac{1}{13^9}$ as 13^{-9}. Last, multiply, yielding $\frac{7^3}{1} \times \frac{1}{13^9} = 7^3 \times 13^{-9}$.
2. Rewrite the expression $\frac{c^4}{d^8}$ as $\frac{c^4}{1} \times \frac{1}{d^8}$. Then use the definition of negative exponents to rewrite the $\frac{1}{d^8}$ as d^{-8}. Last, multiply, yielding $\frac{c^4}{1} \times \frac{1}{d^8} = c^4 \times d^{-8}$.

The table below shows a summary of the exponent rules from this section.

Rule	Example	In Words
Product Rule $a^n a^m = a^{n+m}$	$8^2 \times 8^5 = 8^7$	A base raised to a power, times the same based raised to another power, is the base raised to the sum of the powers.
Quotient Rule $\frac{a^n}{a^m} = a^{(n-m)}$	$\frac{11^{14}}{11^{12}} = 11^{12}$	A base raised to a power, divided by the same based raised to another power, is the base raised to the difference of the powers.
Zero Power Rule $a^0 = 1$ provided that $a \neq 1$	$412^0 = 1$	Any non-zero number raised to the zeroth power equals 1.
Distributive Rule, Multiplication $(a \times b)^n = a^n \times b^n$	$(14 \times 31)^9 = 14^9 \times 31^9$	Exponents distribute across multiplication.
Distributive Rule, Division $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{62}{91}\right)^8 = \frac{62^8}{91^8}$	Exponents distribute across division.
Power Rule $(a^n)^m = a^{(n \times m)}$	$(5^9)^{15} = 5^{135}$	A base raised to a power, raised to another power, is the base raised to the first power times the second power.
Negative Exponent Rule $a^{-n} = \frac{1}{a^n}$ provided that $a \neq 0$	$6^{-8} = \frac{1}{6^8}$ $\frac{1}{12^7} = 12^{-7}$	A base raised to a negative exponent is 1 divided by the base raised to the positive power, and vice versa.

These rules often occur in tandem with each other, but it requires that you carefully apply the rules.

Example 10 **Simplifying Expressions Using Exponent Rules**

Simplify the following:

1. $\left(\frac{4^2 \times 7}{9^3}\right)^5$
2. $\left(\frac{5a^4}{b^9}\right)^6$

Solution

1. **Step 1:** To simplify this, we start by distributing the power 5 across the quotient:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{(4^2 \times 7)^5}{(9^3)^5}$$

Step 2: We distribute the power 5 in the numerator across that multiplication:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5} = \frac{\left(4^2\right)^5 \times 7^5}{\left(9^3\right)^5}$$

Step 3: We apply the power rule where indicated:

$$\left(\frac{4^2 \times 7}{9^3}\right)^5 = \frac{\left(4^2 \times 7\right)^5}{\left(9^3\right)^5} = \frac{\left(4^2\right)^5 \times 7^5}{\left(9^3\right)^5} = \frac{4^{(2\times5)} \times 7^5}{9^{(3\times5)}} = \frac{4^{10} \times 7^5}{9^{15}}$$

1. **Step 1:** To simplify this, we start by distributing the power 6 across the quotient:

$$\left(\frac{5a^4}{b^9}\right)^9 = \frac{\left(5 \times a^4\right)^6}{\left(b^9\right)^6}$$

Step 2: We distribute the power 5 in the numerator across that multiplication:

$$\frac{\left(5 \times a^4\right)^6}{\left(b^9\right)^6} = \frac{(5)^6 \times \left(a^4\right)^6}{\left(b^9\right)^6}$$

Step 3: We apply the power rule where indicated:

$$\frac{(5)^6 \times \left(a^4\right)^6}{\left(b^9\right)^6} = \frac{5^6 a^{24}}{b^{54}}$$

VIDEO

Simplifying Expressions with Exponents

Key Terms

- base
- exponent

Key Concepts

- Exponents are used to express multiplying a number by itself a number of times. The number being multiplied by itself is the base. The number of times it is multiplied by itself is the exponent, which is often referred to as the power.
- Understanding that exponents represent repeated multiplication of a base makes it possible to establish some rules for combining exponential expressions, using the product rule, the quotient rule, and the power rule. Additionally, it allows us to formulate distributive rules for exponents.
- Any non-zero number raised to the 0th power is 1. This makes the definition of the 0th power consistent with the division rule for exponents.

- For consistency, negative exponents represent the reciprocal of the base raised to the power, so that $a^{-n} = \frac{1}{a^n}$, provided that $a \neq 0$.

Formulas

- $$a^n a^m = a^{n+m}$$
- $$\frac{a^n}{a^m} = a^{(n-m)}$$
- $a^0 = 1$, provided that $a \neq 0$
- $$(a \times b)^n = a^n \times b^n$$
- $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
- $$(a^n)^m = a^{(n \times m)}$$
- $a^{-n} = \frac{1}{a^n}$, provided that $a \neq 0$

Videos

- Exponential Notation
- Product and Quotient Rule for Exponents
- Fraction Raised to a Power
- Simplifying Expressions with Exponents

3.9 Scientific Notation

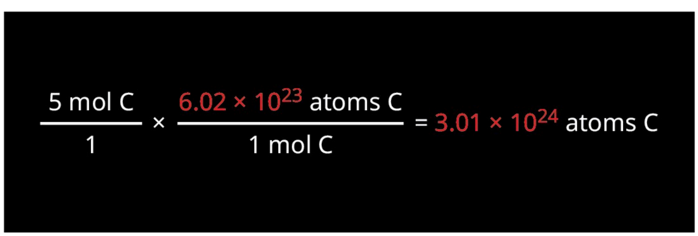


Figure 45: Calculations in the sciences often involve numbers in scientific notation form.

Learning Objectives

After completing this section, you should be able to:

1. Write numbers in standard or scientific notation.
2. Convert numbers between standard and scientific notation.
3. Add and subtract numbers in scientific notation.
4. Multiply and divide numbers in scientific notation.
5. Use scientific notation in computing real-world applications.

The amount of information available on the Internet is simply incomprehensible. One estimate for the amount of data that will be on the Internet by 2025 is 175 Zettabytes. A single zettabyte is one billion trillion. Written out, it is 1,000,000,000,000,000,000,000. One estimate is that we're producing 2.5 quintillion bytes of data per day. A quintillion is a trillion trillion, or, written out, 1,000,000,000,000,000,000. To determine how many days it takes to increase the amount of information that is on the Internet by 1 zettabyte, divide these two numbers, a zettabyte being 1,000,000,000,000,000,000,000, and 2.5 quintillion, being 2,500,000,000,000,000,000, shows it takes 400 days to generate 1 zettabyte of information. But doing that calculation is awkward with a calculator. Keeping track of the zeros can be tedious, and a mistake can easily be made.

On the other end of the scale, a human red blood cell has a diameter of 7.8 micrometers. One micrometer is one millionth of a meter. Written out, 7.8 micrometers is 0.0000078 meters. Smaller still is the diameter of a virus, which is about 100 nanometers in diameter, where a nanometer is a billionth of a meter. Written out, 100 nanometers is 0.0000001 meters. To compare that to engineered items, a single transistor in a computer chip can be 14 nanometers in size (0.000000014 meters). Smaller yet is the diameter of an atom, at between 0.1 and 0.5 nanometers.

Sometimes we have numbers that are incredibly big, and so have an incredibly large number of digits, or sometimes numbers are incredibly small, where they have a large number of digits after the decimal. But using those representations of the names of the sizes makes comparing and computing with these numbers problematic. That's where scientific notation comes in.

Writing Numbers in Standard or Scientific Notation Form

When we say that a number is in **scientific notation**, we are specifying the form in which that number is written. That form begins with an integer with an absolute value between 1 and 9, then perhaps followed the decimal point and then some more digits. This is then multiplied by 10 raised to some power. When the number only has one non-zero digit, the scientific notation form is the digit multiplied by 10 raised to an exponent. When the number has more than one non-zero digit, the scientific notation form is a single digit, followed by a decimal, which is then followed by the remaining digits, which is then multiplied by 10 to a power.

The following numbers are written in scientific notation:

$$1.45 \times 10^3$$

$$-8.345 \times 10^{-4}$$

$$3 \times 10^2$$

$$3.14159 \times 10^0$$

The following numbers are not written in scientific notation:

1.45 because it isn't multiplied by 10 raised to a power

-50.053×10^7 because the absolute value of -50.053 is not at least 1 and less than 10

41.7×10^9 because 41.7 is not at least 1 and less than 10

0.036×10^{-3} because 0.036 is not at least one and less than 10

Example 1 **Identifying Numbers in Scientific Notation**

Which of the following numbers are in scientific notation? If the number is not in scientific notation, explain why it is not.

1. -9.67×10^{20}
2. 145×10^{-8}
3. 1.45

Solution

1. The number -9.67×10^{20} is in scientific notation because the absolute value of -9.67 is at least 1 and less than 10.
2. The number 145×10^{-8} is not in scientific notation because 145 is not at least 1 and less than 10.
3. The number 1.45 is not in scientific notation form. Even though it is at least 1 but less than 10, it is not multiplied by 10 raised to a power.

Some numbers are so large or so small that it is impractical to write them out fully. Avogadro's number is important in chemistry. It represents the number of units in 1 mole of any substance. The substance many be electrons, atoms, molecules, or something else. Written out, the number is: 602,214,076,000,000,000,000,000. Another example of a number that is impractical to write out fully is the length of a light wave. The wavelength of the color blue is about 0.000000450 to 0.000000495 meters. Such numbers are awkward to work with, and so scientific notation is often used. We need to discuss how to convert numbers into scientific notation, and also out of scientific notation.

Recall that multiplying a number by 10 adds a 0 to the end of the number or moves the decimal one place to the right, as in $43 \times 10 = 430$ or $3.89 \times 10 = 38.9$. And if you multiply by 100, it adds two zeros to the end of the number or moves the decimal two places to the right, and so on. For example, $38 \times 100,000 = 3,800,000$ and $32.998 \times 10,000 = 329,980$. Multiplying a number by 1 followed by some number of zeros just adds that many zeros to the end of the number or moves the decimal place that many places to the right. Numbers written as 1 followed by some zeros are just powers of 10, as in $10^1 = 10$, $10^2 = 100$, $10^3 = 1,000$, etc. Generally, $10^n = 1\underbrace{0\ldots0}_{n}$.

We can use this to write very large numbers. For instance, Avogadro's number is 602,214,076,000,000,000,000,000, which can be written as $6.02214076 \times 10^{23}$. The multiplication moves the decimal 23 places to the right.

Similarly, when we divide by 10, we move the decimal one place to the left, as in $\frac{46.7}{10} = 4.67$. If we divide by 100, we move the decimal two places to the left, as in $\frac{3.456}{100} = 0.03456$. In general, when you divide a number by a 1 followed by n zeros, you move the decimal n places to the left, as in $\frac{8,244.902}{1,000,000} = 0.008244902$. This denominator could be written as 10^6. If we use that in the expression and allow for negative exponents, rewrite the number as $\frac{8,244.902}{1,000,000} = \frac{8,244.902}{10^6} =$

$8,244.902 \times 10^{-6}$. With this, we can write division by a 1 followed by n zeros as multiplication by 10 raised to $-n$.

Using that information, we can demonstrate how to convert from a number in standard form into scientific notation form.

Case 1: The number is a single-digit integer.

In this case, the scientific notation form of the number is $digit \times 10^1$.

Case 2: The absolute value of the number is less than 1.

Follow the process below.

- **Step 1:** Count the number of zeros between the decimal and the first non-zero digit. Label this n.
- **Step 2:** Starting with the first non-zero digit of the number, write the digits. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal after the first digit from Step 2.
- **Step 4:** Multiply the number from Step 3 by 10^{n+1}.

  Case 3: The absolute value of the number is 10 or larger.

  Follow the process below.

  - **Step 1:** Count the number of digits that are to the left of the decimal point. Label this n.
- **Step 2:** Write the digits of the number without the decimal place, if one was present. If the number was negative, include the negative sign.
- **Step 3:** If there is more than one digit, place the decimal point after the first digit.
- **Step 4:** Multiply the number from Step 3 by 10^{n-1}.

Example 2 **Writing a Number in Scientific Notation**

Write the following numbers in scientific notation form:

1. 428.9
2. -0.00000981
3. 8

Solution

1. Since the absolute value of 428.9 is 10 or larger, so we use the process from Case 3, above. **Step 1:** There are three digits to the left of the decimal point, so $n = 3$.

Step 2: Write the digits of the number without the decimal place, which is 4289.

Step 3: Since there is more than one digit, place the decimal point after the first digit. We now have 4.289.

Step 4: Since $n = 3$, we multiply 4.289 by 10 raised to the second power, 4.289×10^2.

The scientific notation form of 428.9 is 4.289×10^2.

1. Since the absolute value of -0.00000981 is less than 1, we use the process from Case 2. **Step 1:** The number of zeros between the decimal and the first non-zero digit is 5, so $n = 5$.

Step 2: We write the non-zero digits, including the negative sign, yielding -981.

Step 3: The decimal gets placed to the right of the first digit, resulting in -9.81.

Step 4: Since $n = 5$, we multiply -9.81 by 10 raised to the fourth power, -9.81×10^{-6}.

The scientific notation form of -0.00000981 is 9.81×10^{-6}.

1. Since 8 is a single-digit integer, apply Case 1. The scientific notation form of 8 is 8×10^{1}.

When we write numbers in scientific notation form, we can manipulate the representation of the number by moving the decimal around, and making an appropriate change to the exponent of the 10. For instance, let's look at 145.8141×10^{8}. If we wanted to move the decimal one place to the left, we'd have to increase the power of 10, as shown here: $145.8141 \times 10^{8} = 14.58141 \times 10^{9}$. Since we moved the decimal one to the left, we balance that with moving the exponent up by one. Similarly, if we move the decimal one place to the right, we have to balance that by moving the exponent one to the left, or subtracting one from the exponent, as shown here: $145.8141 \times 10^{8} = 1458.141 \times 10^{7}$. Generally, for a number in the form *number* $\times 10^{n}$:

- If you move the decimal to the left by k digits, you increase the exponent by k.

- If you move the decimal to the right by k digits, you decrease the exponent by k digits.

Example 3 **Increasing the Exponent**

Change 456.142×10^{5} by moving the decimal two places to the left.

Solution

Since we are moving the decimal to the left by two places, we increase the exponent of 10 by 2, so that the exponent is now 7. This gives us $456.142 \times 10^{5} = 4.56142 \times 10^{7}$.

Example 4 **Decreasing the Exponent**

Change 12.3×10^{2} by moving the decimal five places to the right.

Solution

Since we are moving the decimal to the right by five places, we decrease the exponent of 10 by 5, so that exponent is now -3. This give us $12.3 \times 10^{2} = 1230000.0 \times 10^{-3}$.

Converting Numbers from Scientific Notation to Standard Form

In the previous section, converting a number from standard form to scientific notation was explored. Now, we explore converting from scientific notation back into standard form. Doing so involves moving the decimal according to the power of the 10. The decimal is moved a number of steps equal to the exponent of the 10. As demonstrated previously, when the exponent of the

10 is negative, the decimal is moved to the left and when the exponent of the 10 is positive, the decimal is moved to the right.

Example 5 **Converting from Scientific Notation to Standard Form**

Convert the following into standard form:

1. 2.78×10^9
2. 9.04×10^{-8}

Solution

1. Since the exponent is positive, the decimal moves nine places to the right, so 2.78×10^9 is $2,780,000,000$.
2. Since the exponent is negative, the decimal moves eight places to the left, so 9.04×10^{-8} is 0.0000000904.

VIDEO

Converting from Standard Form to Scientific Notation Form

Converting from Scientific Notation Form to Standard Form

TECH CHECK

Scientific Notation on a Calculator

Most scientific and graphing calculators come with the ability to directly convert from standard form to scientific notation. On the TI-83, it is accessed through the MODE menus. For a commonly used, free phone scientific calculator, the calculator can be forced to work in scientific notation mode through its settings.

Some calculators, such as the Desmos online calculator, display scientific notation as a number times 10 to a power as you've seen in this section. However, some calculators indicate scientific notation by replacing the $\times 10^n$ with an E (or EE) followed by the exponent. For example, shows what you may see on a TI-84.

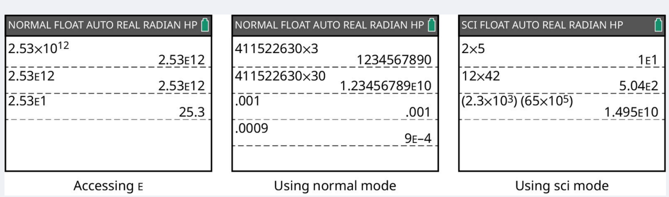


Figure 46: Calculator screens

Adding and Subtracting Numbers in Scientific Notation

To add or subtract numbers in scientific notation, the numbers first need to have the same exponent for the 10s. It is possible to add the following since the powers of 10 match: $4.5 \times 10^4 + 3.15 \times 10^4 = 7.65 \times 10^4$

Notice that the number parts were added, but the exponent part remained the same. This is due to the distributive property of the real numbers. The 10^4 is factored from the two terms, as shown: $4.5 \times 10^4 + 3.15 \times 10^4 = (4.5 + 3.15) \times 10^4 = 7.65 \times 10^4$

Numbers in scientific notation can be added or subtracted directly using a calculator. Simply enter the values in scientific form and set your calculator to display scientific notation.

Example 6 **Adding and Subtracting Numbers in Scientific Notation with the Same Powers of 10**

Calculate the following:

1. $3.8 \times 10^{-3} + 1.006 \times 10^{-3}$
2. $9.61 \times 10^8 - 3.85 \times 10^8$

Solution

1. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^{-3} from the numbers. We then add the number parts separately to get 4.806. $3.8 \times 10^{-3} + 1.006 \times 10^{-3} = (3.8 + 1.006) \times 10^{-3} = 4.806 \times 10^{-3}$
2. Since the powers of 10 match, we use the distributive property of real numbers to factor 10^8 from the numbers. We then subtract the number parts separately to get 5.76.

$$9.61 \times 10^8 - 3.85 \times 10^8 = (9.61 - 3.85) \times 10^8 = 5.76 \times 10^8$$

Adding and subtracting in scientific notation is straightforward when the exponents are the same. There are two issues that can arise. The first issue is what to do if after adding or subtracting the result is not in scientific notation.

Example 7 **Correcting an Answer to Scientific Notation After Adding or Subtracting**

Calculate the following:

1. $7.03 \times 10^{13} + 8.5 \times 10^{13}$
2. $4.3 \times 10^{21} - 4.613 \times 10^{21}$

Solution

1. Since the powers of 10 match, we add the number parts and multiply that by 10^{13}: $7.03 \times 10^{13} + 8.5 \times 10^{13} = (7.03 + 8.5) \times 10^{13} = 15.53 \times 10^{13}$.

However, 15.53×10^{13} is not in scientific notation because the absolute value of 15.53 is more than 10. To put this number in scientific notation, the decimal needs to move one to the left. To balance that move, the power of 10 must be increased by 1. So, the answer in scientific notation is 1.553×10^{14}.

1. Since the powers of 10 match, we add the number parts: $4.3 \times 10^{21} - 4.613 \times 10^{21} = (4.3 - 4.613) \times 10^{21} = -0.313 \times 10^{21}$

However, -0.313×10^{21} is not in scientific notation because it is less than 1. To put it in scientific notation, the decimal needs to move one to the right. To balance that move, the power of 10 must be decreased by 1. So, the answer in scientific notation is -3.13×10^{20}.

The second issue that might be encountered when adding or subtracting is that the powers of 10 do not match. In that case, one of the numbers must be changed so that the powers of 10 match. It is easiest to make the smaller power of 10 larger to match the other power of 10.

For example, to perform the following, $4.5 \times 10^5 + 3.9 \times 10^3$, we'd change the 3.9×10^3 so that the power of 10 is 5. To do so, we need to increase the power of 10 and move the decimal in the number part two places to the left. That would alter 3.9×10^3 into 0.039×10^5. We would use 0.039×10^5 in the addition problem, so that the exponents match, allowing the addition to occur. $4.5 \times 10^5 + 3.9 \times 10^3 = 4.5 \times 10^5 + 0.039 \times 10^5 = (4.5 + 0.039) \times 10^5 = 4.539 \times 10^5$

The steps to take when the exponents of the 10s are not equal are:

Step 1: Increase the smaller exponent to equal the larger exponent. Label the amount increased as n.

Step 2: For the number with the smaller power of 10, move the decimal point of the number part to the left n places.

Step 3: Perform the addition or subtraction.

Step 4: If the result is not in scientific notation, adjust the number to be in scientific notation.

Example 8 **Adding Numbers in Scientific Notation with Different Powers of 10**

Calculate the following:

$$6.1 \times 10^4 + 4.8 \times 10^5$$

Solution

Step 1: The lower exponent is 4. To make this equal to the larger exponent, we increased it by 1.

Step 2: Since the smaller exponent was increased by 1, move the decimal one to the left, so the addition become $6.1 \times 10^4 + 4.8 \times 10^5 = 0.61 \times 10^5 + 4.8 \times 10^5$.

Step 3: Now add the numbers, $0.61 \times 10^5 + 4.8 \times 10^5 = 5.41 \times 10^5$

Step 4: The result is in scientific notation, so no additional adjustment is necessary.

$$6.1 \times 10^4 + 4.8 \times 10^5 = 5.41 \times 10^5$$

Example 9 **Subtracting Numbers in Scientific Notation with Different Powers of 10**

Calculate the following:

$7.9 \times 10^{-15} - 6.8 \times 10^{-13}$

Solution

Step 1: The lower exponent is -15 and the larger is -13. To make -15 equal to the larger exponent, we increased it by 2.

Step 2: Since the smaller exponent increased by 2, move the decimal two to the left. The subtraction changes to $7.9 \times 10^{-15} - 6.8 \times 10^{-13} = 0.079 \times 10^{-13} - 6.8 \times 10^{-13}$.

Step 3: Subtract the numbers, $0.079 \times 10^{-13} - 6.8 \times 10^{-13} = -6.721 \times 10^{-13}$.

Step 4: The result is in scientific notation, so no additional adjustment is necessary.
$7.9 \times 10^{-15} - 6.8 \times 10^{-13} = -6.721 \times 10^{-13}$

Multiplying and Dividing Numbers in Scientific Notation

Multiplying and dividing numbers in scientific notation is somewhat easier than adding or subtracting, because the exponents of the 10s do not have to match. However, it is much more likely that the result will not be in scientific notation, and so that will have to be adjusted at the end. Generally, we multiply or divide the number parts of the two values, and then apply exponent rules to the 10 raised to the powers.

To multiply two numbers in scientific notation:

Step 1: Multiply the number parts.

Step 2: Add the exponents of the 10s.

Step 3: The result is the answer from Step 1 times 10 raised to the answer from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

Example 10 **Multiplying Numbers in Scientific Notation**

Calculate the following:

1. $$\left(4.3 \times 10^{3}\right) \times \left(1.8 \times 10^{7}\right)$$
2. $$\left(5 \times 10^{-1}3\right) \times \left(7.3 \times 10^{6}\right)$$

Solution

1. **Step 1:** Multiply the number parts to get $4.3 \times 1.8 = 7.74$.

Step 2: Add the exponents of the 10s to get $3 + 7 = 10$.

Step 3: The result is then 7.74×10^{10}.

Step 4: This number is already in scientific notation, so no additional adjustment is necessary, $\left(4.3 \times 10^{3}\right) \times \left(1.8 \times 10^{7}\right) = 7.74 \times 10^{10}$.

1. **Step 1:** Multiply the number parts to get $5 \times 7.3 = 36.5$.

Step 2: Add the exponents of the 10s to get $-13 + 6 = -7$.

Step 3: The result then is 36.5×10^{-7}.

Step 4: Since the number is not in scientific notation, it must be adjusted. To put 36.5×10^{-7} into scientific notation, the decimal moves one to the left, so the exponent would be increased by 1, giving 3.65×10^{-6}.

$$\left(5 \times 10^{-13}\right) \times \left(7.3 \times 10^{6}\right) = 3.65 \times 10^{-6}$$

VIDEO

Multiplying Numbers in Scientific Notation

Dividing Numbers in Scientific Notation

To divide two numbers that are in scientific notation:

Step 1: Divide the number parts.

Step 2: Subtract the exponent of the denominator from the exponent of the numerator.

Step 3: The answer is the result from Step 1 times 10 raised to the result from Step 2.

Step 4: If the number is not in scientific notation, adjust it appropriately.

Example 11 **Dividing Numbers in Scientific Notation**

Calculate the following:

1. $\left(8.4 \times 10^{31}\right)$-$\left(2.1 \times 10^{7}\right)$
2. $\left(4.14 \times 10^{-13}\right)$-$\left(8.28 \times 10^{9}\right)$

Solution

1. $$\left(8.4 \times 10^{31}\right)\text{-}\left(2.1 \times 10^{7}\right)$$

Step 1: Divide the number parts to get $8.4 \div 2.1 = 4$.

Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $37 - 7 = 24$.

Step 3: The result is then 4×10^{24}.

Step 4: This number is already in scientific notation, so no adjustment is necessary. $\left(8.4 \times 10^{31}\right)$-$\left(2.1 \times 10^{7}\right) = 4 \times 10^{24}$

1. $$\left(4.14 \times 10^{-13}\right)\text{-}\left(8.28 \times 10^{9}\right)$$

Step 1: Divide the number parts to get $4.14 \div 8.28 = 0.5$.

Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $-13 - 9 = -22$.

Step 3: The result then is 0.5×10^{-22}.

Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.5×10^{-22} into scientific notation, the decimal needs to move one to the right, so the exponent is decreased by 1, giving 5×10^{-23}. $\left(4.14 \times 10^{-13}\right)$-$\left(8.28 \times 10^{9}\right) = 5 \times 10^{-23}$

VIDEO

Dividing Numbers in Scientific Notation

Using Scientific Notation in Computing Real-World Applications

As noted at the start of this section, scientific notation is useful when the standard representation of a number is awkward or impractical, which occurs when the numbers being used are extremely large or extremely small. For example, Venus is 67,667,000 miles from the sun. In scientific notation, this is 6.7667×10^{7}. Planetary and galaxy distances is one set of numbers that is easier to express using scientific notation.

Example 12 **Calculating Distances**

How much farther from the sun is Earth compared to Venus if Venus is 6.7667×10^{7} miles from the sun and Earth is 9.1692×10^{7} miles from the sun?

Solution

To determine how much farther Earth is compared to Venus, we'd subtract the distances. $9.1692 \times 10^{7} - 6.7667 \times 10^{7} = 2.4025 \times 10^{7}$.

So, Earth is 2.4025×10^{7} miles farther from the sun than Venus.

Example 13 **Calculating Probability**

The probability of winning the Mega Millions lottery is published as 3.304693×10^{-9}. The probability of being hit by lightning is approximated to be 2×10^{-6}. How many times more likely are you to be hit by lightning than win the Mega Millions?

Solution

To find out how many times more likely you are to be hit by lightning, divide the probability of being hit by lightning by the probability of winning the Mega Millions.

$$\left(2 \times 10^{-6}\right)\text{-}\left(3.304693 \times 10^{-9}\right) = 6.609386 \times 10^{3}$$

Step 1: Divide the number parts to get $= 0.6052$ (rounded to the fourth digit).

Step 2: Subtract the exponent of the denominator from the exponent of the numerator to get $-6 - (-9) = 3$.

Step 3: The result then is 0.6052×10^{3}.

Step 4: Since this number is not in scientific notation, it must be adjusted. To put 0.6052×10^{3} into scientific notation, the decimal needs to move one place to the right, so the exponent is decreased by 1, giving 6.052×10^{2}.

You are 6.052×10^2, or 605.2, times more likely to be hit by lightning than you are to win the Mega Millions.

Example 14 **Calculating Time and Length**

Sometimes it is entertaining to determine the time it takes for something to happen. Fingernails grow about 8.032×10^{-11} km per minute. How many kilometers long would fingernails be after 6×10^4 minutes?

Solution

To find the length of the fingernails after the specified time, we multiply their rate of growth and the time they've grown. $\left(8.032 \times 10^{-11}\right) \times \left(6 \times 10^4\right) = 48.192 \times 10^{-7} = 4.8192 \times 10^{-6}$

So, after 6×10^4 minutes, the fingernails would be 4.8192×10^{-6} km long. To put this in perspective, 1×10^{-6} km is a millimeter, and 6×10^4 minutes is about 4.16 days. So, after about 4.16 days, fingernails have grown about 4.8 millimeters.

Example 15 **Calculating Data Generated**

As mentioned in the opening to this section, it is estimated that we're producing 2.5 quintillion bytes of data per day. A good estimate is that there are 7.674 billion people on the planet. Convert both of those numbers to scientific notation, and then determine how much data is being generated per person each day.

Solution

Written in standard form, 2.5 quintillion is 2,500,000,000,000,000,000. Changing that to scientific notation, move the decimal 18 places, so 2.5 quintillion bytes $= 2.5 \times 10^{18}$ bytes. Writing 7.647 billion in scientific notation would be 7.647×10^9 because a billion is $1{,}000{,}000{,}000 = 10^9$. So, to find out how much data is being produced daily per person, we would divide these two numbers. $\frac{2.5 \times 10^{18}}{7.647 \times 10^9} = 0.327 \times 10^9 = 3.27 \times 10^8$

In standard form, that's 327,000,000 bytes per person, so 327 million bytes of data daily are being produced per person.

VIDEO

Application of Scientific Notation

What Numbers Could Be Considered "Too Big" or "Too Small"?

One wonders when the numbers we represent become too large or small for consideration. Perhaps the following examples put limits on what is meaningful. The number of particles in the known universe has been estimated at 4×10^{80} particles. The smallest distance that has been measured is 1×10^{-18} m, though the theoretical smallest measurable value is 1×10^{-35} m. The distance across the universe is 4.4×10^{26} m. Considering what those numbers represent, the extreme largest and extreme smallest, they might be numbers that constrain what we should reasonably be expected to deal with.

Key Terms

- scientific notation
- standard notation

Key Concepts

- Some numbers are so large or so small that writing the number out is clumsy and make it difficult to determine the true size of the number. Scientific notation makes the number more readable and make the relative size of the number immediately apparent.
- A number written in scientific notation is a number at least 1 and smaller than 10 multiplied by 10 raised to an exponent. Converting between scientific notation and standard notation involves correctly applying multiplication and division by powers of 10, which in practice equates to understanding how moving the decimal point of a number impacts the exponent of 10.
- Adding and subtracting numbers in base 10 requires the exponent of 10 in each number be the same. Once the numbers are converted to have the same exponent with the ten, then the numbers are added or subtracted as indicated, with the power of 10 remaining the same. If the result is not in scientific notation (for instance, the number has exceeded 10), then then number must be converted into scientific notation.
- Multiplying and dividing numbers in scientific notation is done by multiplying or dividing the number parts, then multiplying or dividing the 10 raised to the power parts, then multiplying those two results. If the new number is not in scientific notation, then the result must be converted into scientific notation.

Videos

- Converting from Standard Form to Scientific Notation Form
- Converting from Scientific Notation Form to Standard Form
- Multiplying Numbers in Scientific Notation
- Dividing Numbers in Scientific Notation
- Application of Scientific Notation

3.10 Arithmetic Sequences

Learning Objectives

After completing this section, you should be able to:

1. Identify arithmetic sequences.
2. Find a given term in an arithmetic sequence.
3. Find the nth term of an arithmetic sequence.
4. Find the sum of a finite arithmetic sequence.

5. Use arithmetic sequences to solve real- world applications

   As we saw in the previous section, we are adding about 2.5 quintillion bytes of data per day to the Internet. If there are 550 quintillion bytes of data today, then there will be 552.5 quintillion bytes tomorrow, and 555 quintillion bytes in 2 days. This is an example of an arithmetic sequence. There are many situations where this concept of fixed increases comes into play, such as raises or table arrangements.

Identifying Arithmetic Sequences

A **sequence** of numbers is just that, a list of numbers in order. It can be a short list, such as the number of points earned on each assignment in a class, such as {10, 10, 8, 9, 10, 6, 10}. Or it can be a longer list, even infinitely long, such as the list of prime numbers. For example, here's a sequence of numbers, specifically, the squares of the first 12 natural numbers.

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}

Each value in the sequence is called a **term**. Terms in the list are often referred to by their location in the sequence, as in the nth term. For the sequence {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}, the first term of the sequence is 1, the fourth term is 16, and so on. In the sequence of assignment scores {10, 10, 8, 9, 10, 6, 10}, the first term is 10 and the third term is 8 .

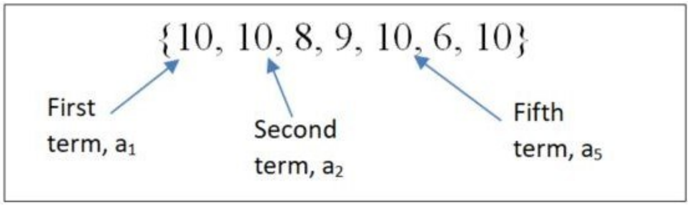


Figure 47: Sequence showing first, second, and fifth terms

The notation we use with sequences is a letter, which represents a term in the sequence, and a subscript, which indicates what place the term is in the sequence. For the sequence {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144}, we will use the letter a as a value in the sequence, and so a_5 would be the term in the sequence at the fifth position. That number is 25, so we can write $a_5 = 25$.

In this section, we focus on a special kind of sequence, one referred to as an **arithmetic sequence**. Arithmetic sequences have terms that increase by a fixed number or decrease by a fixed number, called the **constant difference** (denoted by d), provided that value is not 0. This means the next term is always the previous term plus or minus a specified, constant value. Another way to say this is that the difference between any consecutive terms of the sequence is always the same value.

To see a constant difference, look at the following sequence: {7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87}. illustrates that each term of the sequence is the previous term plus 8. Eight is the constant difference here.

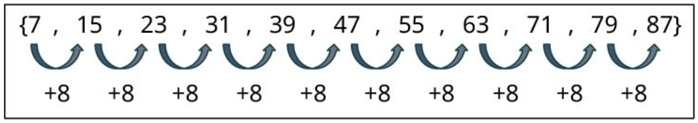


Figure 48: Sequence of numbers with 8 added to each term

Example 1 **Identifying Arithmetic Sequences**

Determine if the following sequences are arithmetic sequences. Explain your reasoning.

1. $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$
2. $\{20, 40, 80, 160, 320, 640\}$
3. $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$

Solution

1. In the sequence $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$, every term is the previous term plus 3. The ellipsis indicates that the pattern continues, which means keep adding 3 to the previous term to get the new term. Therefore, this is an infinite arithmetic sequence.
2. In the sequence $\{20, 40, 80, 160, 320, 640\}$, terms increase by various amounts, for instance from term 1 to term 2, the sequence increases by 20, but from term 2 to term 3 the sequence increases by 40. So, this is not an arithmetic sequence.
3. In the sequence $\{7, 1, -5, -11, -17, -23, -29, -34, -40\}$, every term is the previous term minus 6, so this is an arithmetic sequence.

Arithmetic sequences can be expressed with a formula. When we know the first term of an arithmetic sequence, which we label a_1, and we know the constant difference, which is denoted d, we can find any other term of the arithmetic sequence. The formula for the i th term of an arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

FORMULA

If we have an arithmetic sequence with first term a_1 and constant difference d, then the i th term of the arithmetic sequence is $a_i = a_1 + d \times (i - 1)$.

Let's examine the formula with this arithmetic sequence: $\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}$. In this sequence $a_1 = 4$ and $d = 3$. The table below shows the values calculated.

i, Place in Sequence	a_i, i^{th} Term	Value of Term	Term Written as $a_1 + 3 \times (i-1)$
1	a_1	4	$4 + 3 \times 0$
2	a_2	7	$4 + 3 \times 1$
3	a_3	10	$4 + 3 \times 2$
4	a_4	13	$4 + 3 \times 3$
5	a_5	16	$4 + 3 \times 4$
i	a_i		$4 + 3 \times (i-1)$

We can see how the i th term can be directly calculated. In this sequence, the formula is $a_1 + 3 \times (i-1)$ where the first term, a_1, is 4 and the constant difference d is 3. We can then determine the 47 th term of this sequence: $a_{47} = 4 + 3 \times (47-1) = 4 + 3 \times 46 = 4 + 138 = 142$.

Example 2 **Calculating a Term in an Arithmetic Sequence**

Identify a_1 and d for the following arithmetic sequence. Use this information to determine the 60 th term.

$$\{18, 31, 44, 57, 70, 83, \ldots\}$$

Solution

Inspecting the sequence shows that $a_1 = 18$ and $d = 13$. We use those values in the formula, with $i = 60$.

$$a_{60} = a_1 + d \times (i-1) = 18 + 13 \times (60-1) = 18 + 13 \times 59 = 18 + 767 = 785$$

VIDEO

Arithmetic Sequences

If we know two terms of the sequence, it is possible to determine the general form of an arithmetic sequence, $a_i = a_1 + d \times (i-1)$.

FORMULA

If we have the ith term of an arithmetic sequence, a_i, and the jth term of the sequence, a_j, then the constant difference is $d = \frac{a_j - a_i}{j-i}$ and the first term of the sequence is $a_1 = a_i - d(i-1)$.

Example 3 **Determining First Term and Constant Difference Using Two Terms**

A sequence is known to be arithmetic. Two of its terms are $a_7 = 56$ and $a_{19} = 104$. Use that information to find the constant difference, the first term, and then the 50 th term of the sequence.

Solution

To find the constant difference, use $d = \frac{a_j - a_i}{j-i}$. The location of the terms is given by the subscript of the two a terms, $i = 7$ and $j = 19$. So, the constant difference can be calculated as such:

$$d = \frac{104 - 56}{19 - 7} = \frac{48}{12} = 4$$

The constant difference of 4 is then used to find a_1.

$$a_1 = a_i - d(i-1) = a_7 - 4(7-1) = 56 - 4 \times 6 = 32$$

So $d = 4$ and $a_1 = 32$.

With this information, the 50 th term can be found.

$$a_{50} = a_1 + d \times (i-1) = 32 + 4 \times (50-1) = 32 + 4 \times 49 = 32 + 196 = 228$$

The 50 th term is $a_{50} = 228$.

VIDEO

Finding the First Term and Constant Difference for an Arithmetic Sequence

Finding the Sum of a Finite Arithmetic Sequence

Sometimes we want to determine the sum of the numbers of a finite arithmetic sequence. The formula for this is fairly straightforward.

FORMULA

The sum of the first n terms of a finite arithmetic sequence, written s_n, with first and last term a_1 and a_n, respectively, is $s_n = n\left(\frac{a_1 + a_n}{2}\right)$.

Example 4 **Finding the Sum of a Finite Arithmetic Sequence**

What is the sum of the first 60 terms of an arithmetic sequence with $a_1 = 4.5$ and $d = 2.5$?

Solution

The formula requires the first and last terms of the sequence. The first term is given, $a_1 = 4.5$. The 60 th term is needed. Using the formula $a_1 = a_i + d(i-1)$ provides the value for the 60 th term.

$$a_{60} = 4.5 + 2.5(60-1) = 4.5 + 2.5 \times 59 = 4.5 + 147.5 = 152$$

Applying the formula $s_n = n\left(\frac{a_1 + a_n}{2}\right)$ provides the sum of the first 60 terms.

$$s_{60} = 60\left(\frac{4.5 + 152}{2}\right) = 60 \times \frac{156.5}{2} = 4,695$$

The sum of the first 60 terms is 4,695.

VIDEO
Finding the Sum of a Finite Arithmetic Sequence

Using Arithmetic Sequences to Solve Real-World Applications

Applications of arithmetic sequences occur any time some quantity increases by a fixed amount at each step. For instance, suppose someone practices chess each week and increases the amount of time they study each week. The first week the person practices for 3 hours, and vows to practice 30 more minutes each week. Since the amount of time practicing increases by a fixed number each week, this would qualify as an arithmetic sequence.

Example 5 **Applying an Arithmetic Sequence**

Jordan has just watched *The Queen's Gambit* and decided to hone their skills in chess. To really improve at the game, Jordan decides to practice for 3 hours the first week, and increase their time spent practicing by 30 minutes each week. How many hours will Jordan practice chess in week 20?

Solution

Jordan's practice scheme is an arithmetic sequence, as it increases by a fixed amount each week. The first week there are 3 hours of practice. This means $a_1 = 3$. Jordan increases the time spent practicing by 30 minutes, or half an hour, each week. This means $d = 0.5$. Using those values, and that we want to know the amount of time Jordan will study in week 20, we determine the time in week 20 using $a_i = a_1 + d \times (i - 1)$.

$$a_{20} = 3 + 0.5 \times (20 - 1) = 3 + 0.5 \times 19 = 3 + 9.5 = 12.5$$

So, Jordan will practice 12.5 hours in week 20.

Example 6 **Finding the Sum of a Finite Arithmetic Sequence**

Let's check back in on Jordan. Recall, Jordan had just watched *The Queen's Gambit* and decided to hone their skills, practicing for 3 hours the first week, and increasing the time spent practicing by 30 minutes each week. How many hours total will Jordan have practiced chess after 30 weeks of practice?

Solution

To calculate the total amount of time that Jordan practiced, we need to use $s_n = n\left(\frac{a_1+a_n}{2}\right)$. The formula requires the first and last terms of the sequence. Since Jordan practiced 3 hours in the first week, the first term is $a_1 = 3$. Because we want the total practice time after 30 weeks, we need the 30 th term. Because the constant difference is $d = 0.5$, the 30 th term is $a_{30} = 3 + 0.5(30 - 1) = 3 + 0.5 \times 29 = 3 + 14.5 = 17.5$.

Applying the formula $s_n = n\left(\frac{a_1+a_n}{2}\right)$ provides the sum of the first 30 terms.

$$s_{30} = 30\left(\frac{3 + 17.5}{2}\right) = 60 \times \frac{20.5}{2} = 615$$

This means that Jordan practiced a total of 615 hours after 30 weeks.

WHO KNEW?

The Fibonacci Sequence

Not all sequences are arithmetic. One special sequence is the **Fibonacci sequence**, which is the sequence that has as its first two terms 1 and 1. Every term thereafter is the sum of the previous two terms. The first nine terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, and 34.

This sequence is found in nature, architecture, and even music! In nature, the Fibonacci sequence describes the spirals of sunflower seeds, certain galaxy spirals, and flower petals. In music, the band Tool used the Fibonacci sequence in the song "Lateralus." The Fibonacci sequence even relates to architecture, as it is closely related to the golden ratio.

VIDEO

Fibonacci Sequence and "Lateralus"

Key Terms

- sequence
- term of a sequence
- arithmetic sequence
- first term
- constant difference

Key Concepts

- A sequence is a list of numbers. Any individual number in that list, or sequence, is a term of the sequence. A specific term of a sequence is denoted by the sequence symbol with a subscript indicating where the term in the sequence is.
- A special form of a sequence is an arithmetic sequence. Each arithmetic sequence is determined by its first term and its constant difference. Any term in an arithmetic sequence is determined by adding the constant difference to the preceding term.
- If the first term and the constant difference of an arithmetic sequence are known, then any term of the sequence can be found directly.
- Because arithmetic sequences follow such a strict pattern, the sum of the first n terms of an arithmetic sequence can be determined with the formula $s_n = n\left(\frac{a_1+a_n}{2}\right)$.

Formulas

- $a_i = a_1 + d \times (i-1)$
- $d = \frac{a_j - a_i}{j-i}$
- $a_1 = a_i - d(i-1)$

- $s_n = n\left(\frac{a_1+a_n}{2}\right)$

Videos

- Arithmetic Sequences
- Finding the First Term and Constant Difference for an Arithmetic Sequence
- Finding the Sum of a Finite Arithmetic Sequence
- Fibonacci Sequence and "Lateralus"

3.11 Geometric Sequences

Figure 49: Savings grows in a geometric sequence.

Learning Objectives

After completing this section, you should be able to:

1. Identify geometric sequences.
2. Find a given term in a geometric sequence.
3. Find the nth term of a geometric sequence.
4. Find the sum of a finite geometric sequence.
5. Use geometric sequences to solve real-world applications.

   One of the concerns when investing is the **doubling time**, which is length of time it takes for the value of the investment to be twice, or double, that of its starting value. A shorter doubling times means the investment gets bigger, sooner. For example, if you invest \$200 in an account with an 8-year doubling time, then in 8 years the value of the account will be double the starting amount, or $2 \times \$200 = \$400$. After another 8 years (for a total of 16 years) the investment would be twice its value after the first 8 years, or $2 \times (2 \times \$400) = 2 \times (\$400) = \$800$. Every 8 years, the investment would double again, so after the third 8-year period, the investment would be worth $2 \times 2 \times (2 \times \$400) = \$1,600$. This process exhibits exponential growth, an application of geometric sequences, which is explored in this section.

Identifying Geometric Sequences

We know what a sequence is, but what makes a sequence a geometric sequence? In an arithmetic sequence, each term is the previous term plus the constant difference. So, you add a (possibly negative) number at each step. In a **geometric sequence**, though, each term is the previous term multiplied by the same specified value, called the **common ratio**. In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$ the common ratio is 2. To see the difference between an arithmetic sequence and geometric sequence, examine these two sequences (Figures 3.52 and 3.53).

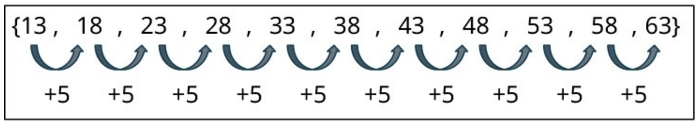


Figure 50: Arithmetic sequence

Each term in this arithmetic sequence is the previous term plus 5.

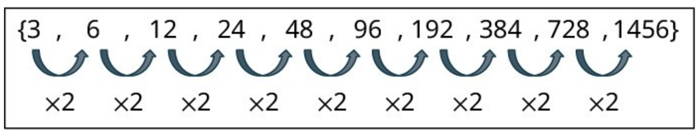


Figure 51: Geometric sequence

Each term in this geometric sequence is the previous term times 2.

In the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, 1456\}$, the numbers get big fairly quickly, and stay positive. However, that's not always the case with geometric sequences. Depending on the value of the common ratio, the terms could increase each time (like in the one shown in), or the terms can get smaller each time, or the terms can alternate between positive and negative values. It all depends on the value of the common ratio, r.

Consider this geometric sequence:

$$\{5, 15, 45, 135, 405, 2025\ldots\}$$

Each term is the previous term times 5, which means the common ratio is 5. This common ratio is larger than 1, and so the terms increase each time. Now, look at this geometric sequence:

$$\{2, -6, 18, -54, 162, -486, 1458\ldots\}$$

Each term is the previous term times -3, and the sign of the terms alternate from positive to negative. Then, there's this geometric sequence:

$$\{9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}\ldots\}$$

Each term is the previous term times $\frac{1}{3}$, and the terms decrease each time. What we should take away from these three examples is if the common ratio is a positive number larger than 1, then the sequence increases. If the common ratio is a negative number, then the sign of the terms alternates between positive and negative. If the common ratio is between 0 and 1, then the terms decrease.

Two special cases of geometric sequences are when the constant ratio is 1 and when the common ratio is 0. When the constant ratio is 1, every term of the sequence is the same, as in $\{3, 3, 3, 3, 3, 3, 3, 3, 3\}$. This is referred to as a **constant sequence**. When the constant ratio is 0, the first term can be any number, but every term after the first term is 0, as in $\{-43.2, 0, 0, 0, 0, 0, 0, 0\}$.

Example 1 **Identifying Geometric Sequences**

For each sequence, determine if the sequence is a geometric sequence. If so, identify the common ratio.

1. $\{5, 20, 80, 320, 1, 280, 5, 120, 20, 480, \ldots\}$
2. $\{-3, 6, -12, 24, 11, 33\}$
3. $\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$

Solution

1. In the sequence $\{5, 20, 80, 320, 1, 280, 5, 120, 20, 480, \ldots\}$, the jump from 5 to 20 is a multiplication by 4, as is the next jump to 80, and the next to 320. Each term is 4 times the previous term. Since each term is 4 times the previous, this is a geometric sequence. The common ratio is 4.
2. In the sequence $\{-3, 6, -12, 24, 11, 33\}$, notice that 6 is -3 times -2. The jump from 6 to -12 is another multiplication by negative. So, if this is a geometric sequence, each term should be the previous term times -2. But the change from 24 to 11 is not a multiplication by -2, This means the sequence is not a geometric sequence.
3. In the sequence $\{4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, the change from 4 to 2 is a multiplication by $\frac{1}{2}$, as is the next jump, from 2 to 1, as is the next from 1 to $\frac{1}{2}$. Each term is $\frac{1}{2}$ times the previous term. Since each term is $\frac{1}{2}$ times the previous, this is a geometric sequence. The common ratio is $\frac{1}{2}$.

As with arithmetic sequences, the first term of a geometric sequence is labeled a_1. The number that is multiplied by each term is called the common ratio and is denoted r. So, if the first term is known, a_1, and the common ratio is known, r, then the n th term, a_n, can be calculated with the formula $a_n = a_1 r^{n-1}$.

FORMULA

The nth term of the geometric sequence, a_n, with first term a_1 and common ratio r, is $a_n = a_1 r^{n-1}$.

Return to the sequence $\{3, 6, 12, 24, 48, 96, 192, 384, 728, \ldots\}$. We observe that the first term is 3, so $a_1 = 3$. We also found that the common ratio is 2, so $r = 2$. The table below shows how any term can be calculated using just a_1 and r.

i, Place in Sequence	a_i, i^{th},Term	Value of Term	Term Written as $a_1 \times r^{i-1}$
1	a_1	3	3×2^0
2	a_2	6	3×2^1
3	a_3	12	3×2^2
4	a_4	24	3×2^3
5	a_5	48	3×2^4
i	a_i		$3 \times 2^{i-1}$

Example 2 **Determining the Value of a Specific Term in a Geometric Sequence**

In the following geometric sequences, determine the indicated term of the geometric sequence with a given first term and common ratio.

1. Determine the 9 th term of the geometric sequence with $a_1 \times 6$ and $r = 3$.
2. Determine the 11 th term of the geometric sequence with $a_1 = 2$ and $r = -5$.

Solution

1. Using $a_n = a_1 r^{n-1}$ with $a_1 = 6$, $r = 3$, and $n = 9$, we calculate

$$a_9 = a_1 r^{9-1} = 6 \times (3)^{9-1} = 6 \times (3)^8 = 6 \times 6561 = 39366$$

. The 9 th term of the geometric sequence with $a_1 = 6$ and $r = 3$ is $a_9 = 39366$.

2. Using $a_n = a_1 r^{n-1}$ with $a_1 = 2$, $r = -5$, and $n = 11$, we calculate

$$a_{11} = a_1 r^{11-1} = 2 \times (-5)^{11-1} = 2 \times (-5)^{10} = 2 \times 9,765,625 = 19,531,250$$

.

VIDEO

Geometric Sequences

Finding the Sum of a Finite Geometric Sequence

As with arithmetic sequences, it is possible to add the terms of the geometric sequence. Like arithmetic sequences, the formula for the finite sum of the terms of a geometric sequence has a straightforward formula.

FORMULA

The sum of the first n terms of a finite geometric sequence, written s_n, with first term a_1 and common ratio r, is $s_n = a_1\left(\frac{1-r^n}{1-r}\right)$ provided that $r \neq 1$.

Example 3 **Calculating the Sum of a Finite Geometric Sequence**

1. What is the sum of the first 13 terms of the geometric sequence with first term $a_1 = 5$ and common ratio $r = 3$?
2. What is the sum of the first 7 terms of the geometric sequence with first term $a_1 = 16$ and common ratio $r = \frac{1}{8}$?

Solution

1. Using $a_1 = 5$, $r = 3$, and $n = 13$, we find that the sum is: $S_{13} = a_1\left(\frac{1-r^{13}}{1-r}\right) = 5\left(\frac{1-3^{13}}{1-3}\right) = 5\left(\frac{1-1{,}594{,}323}{-2}\right)$
   $= 5\left(\frac{-1{,}594{,}322}{-2}\right) = 5(797,161) = 3,985,805$ The sum of the first 13 terms of this geometric sequence is 3,985,805.
2. Using $a_1 = 16$, $r = \frac{1}{8}$, and $n = 7$, we find that the sum is:

$$s_7 = a_1\left(\frac{1-r^7}{1-r}\right) = 16\left(\frac{1-\left(\frac{1}{8}\right)^7}{1-\left(\frac{1}{8}\right)}\right) = 16\left(\frac{1-\left(\frac{1}{2{,}097{,}152}\right)}{\frac{7}{8}}\right)$$

$$= 16\left(\frac{\frac{2{,}097{,}151}{2{,}097{,}152}}{\frac{7}{8}}\right) = 16\left(\frac{299{,}593}{262{,}144}\right) = \frac{299{,}593}{16{,}384} = 18.2857$$

The sum of the first 7 terms of this geometric sequence is 18.2857.

Using Geometric Sequences to Solve Real-World Applications

Geometric sequences have a multitude of applications, one of which is compound interest. Compound interest is something that happens to money deposited into an account, be it savings or an individual retirement account, or IRA. The interest on the account is calculated and added to the account at regular intervals. This means the interest that was earned later gains its own interest. This allows the money to grow faster. If that interest is added every month, we say it is compounded monthly. If the interest is added daily, then we say it is compounded daily. The amount of money that is deposited into the account is called the principal and is denoted P. The account earns money on that principal. The amount it earns is a percentage of the money in the account. The interest rate, expressed as a decimal, is denoted r.

FORMULA

If you deposit P dollars in an account that earns interest compounded yearly, then the amount in the account, A, after t years is calculated with the formula: $A = P(1 + r)^t$. This is a geometric sequence, with constant ratio $(1 + r)$ and first term $a_1 = P$.

Example 4 **Calculating Interest Compounded Yearly**

Daryl deposits $1,000 in an account earning 4% interest compounded yearly. How much money is in the account after 25 years?

Solution

Using $A = P(1+r)^t$ with $P = 1000$, $r = 0.04$, and $t = 25$, we find that $A = P(1+r)^t = 1,000 \times (1+0.04)^{25} = 1,000 \times (1.04)^{25} = 1,000 \times 2.66583633 = 2,665.85$. After 25 years, there is $\$2,665.84$ in the account.

Another application of geometric sequences is exponential growth. This arises in biology quite frequently, especially in relation to bacterial cultures, but also with other organism population models. In bacterial cultures, the time it takes the population to double is often recorded. This time to double is the same, regardless of how big the population gets. So, if the population doubles after 3 hours, it doubles again after another 3 hours, and again after another 3 hours, and so on. Put into geometric sequence language, it has a common ratio of 2.

Example 5 **Doubling a Bacterial Culture**

When *Escherichia coli* (*E. coli*) is in a broth culture at 37°C, the population of *E. coli* doubles in number with 30 organisms, how many *E. coli* bacteria are present in the culture after 16 hours?

Solution

Since the population is doubling every 20 minutes, this is a geometric sequence situation with common ratio $r = 2$. The culture begins with 30 organisms, so $a_1 = 30$. The time,16 hours, is 48 twenty-minute periods, so we're looking for the 48th term in the sequence. Using these values in the geometric sequence formula gives

$$a_{48} = a_1 r^{n-1} = 30 \times 2^{48-1} = 30 \times 2^{47} = 30 \times \left(1.40737 \times 10^{14}\right) = 4.22212 \times 10^{15}$$

.

So, after 16 hours, the culture contains 4.22212×10^{15} *E. coli* organisms. That's more than 4,000 trillion bacteria.

Example 6 **Applying the Sum of a Finite Geometric Sequence**

A player places one grain of rice on the first square of a chess board. On the second square, the player places 2 grains of rice. On the third square, the player places 4 grains of rice. On each successive square of the board, the player doubles the number of grains of rice placed on the chess board. When the player places the last rice on the 64th square, how many total grains of rice have been placed on the board?

Solution

Since the number of grains of rice is doubled at each step, this is a geometric sequence with first term $a_1 = 1$ and common ratio $r = 2$. Rice is placed on 64 total squares, so we want the sum of the first 64 terms. Using this information and the formula, the total number of grains of rice on the board will be:

$$s_{64} = a_1\left(\frac{1-r^{n-1}}{1-r}\right) = 1 \times \left(\frac{1-2^{64-1}}{1-2}\right) = \left(\frac{1-2^{63}}{-1}\right) = -\left(1-2^{63}\right)$$
$$= -\left(-9.2233720369 \times 10^{18}\right) = 9.2233720369 \times 10^{18}$$

That's a 20-digit number!

VIDEO

Sum of a Finite Geometric Sequence

Key Terms

- geometric sequence
- common ratio

Key Concepts

- A special form of a sequence is a geometric sequence. Each geometric sequence is determined by its first term and its constant ratio. Any term in a geometric sequence is determined by multiplying the constant ratio to the preceding term.
- If the first term and the constant ratio of a geometric sequence are known, then any term of the sequence can be found directly.
- Because geometric sequences follow such a strict pattern, the sum of the first n terms of a geometric sequence can be determined with the formula $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$.
- Finding the sum of a finite geometric sequence
- Applying arithmetic sequences

Formulas

- $a_n = a_1 r^{n-1}$
- $s_n = a_1\left(\frac{1-r^{n-1}}{1-r}\right)$

Videos

- Geometric Sequences
- Sum of a Finite Geometric Sequence

Encryption Throughout History

Encryption began at least as far back as the Roman Empire. During the reign of Caesar, a particular cypher was used, fittingly named the Caesar Cypher. This encryption process granted the Romans a great tactical advantage. Even if a message was intercepted, it would not make sense to the person intercepting the message.

Find four instances when encryption was used and cracked over the course of history.

Projects

The Golden Ratio in Art and Architecture

The golden ratio has been used in art and architecture as far back as ancient Greece (possibly further). It also appears in South America (Incan architecture). Find five instances of the use of the golden ratio in art or architecture and describe its use in each of those instances.

Your Budget

Budgeting either is, or will shortly be, an important aspect of your life. Managing money well reduces stress in your life, and provides space for planning for future expenses, such as vacations or home improvements.

Imagine your life 10 years from now. Estimate your monthly income. Identify expenses you will encounter monthly (mortgage or rent, car payment, insurance, entertainment, etc.). Decide on an amount you plan to save monthly (this is treated as an expense). Create a spreadsheet with those values. Record your monthly net income (your income minus your expenses). Determine how much money you will have saved over the course of 5 years (ignore interest). Write a reflection on your anticipated financial health.

Estimating Pi

The value of pi is the ratio of the circumference of a circle to the diameter of the circle. It is also equal to the ratio of the area of the circle to the square of the radius of the square.

Research three ways to physically estimate pi.

Estimate pi using all three processes you found.

Present your process and solutions in class.

Design Your Own Shift Cypher

A cypher is a message written in such a way as to mask its contents. Changing a message into its cypher form is called encryption. Decryption or deciphering is the process of changing a cyphertext message back into the original (legible) message. One process of encryption is to scramble the letters, symbols, and punctuation of a message according to a mathematical rule. One rule that could be used for such a cypher is addition in a chosen modulus. In this project, you will create such a cypher, encrypt a message, and then decrypt the message.

Step 1: Choose the letters, symbols, and punctuation marks you want to allow in your messages. This should include at least the uppercase letters and a space character. This is your character set.

Step 2: Count the number of characters you will use. Label this number n.

Step 3: Pair each character of your character set an integer from 0 to $(n-1)$. Do not assign more than one character to an integer.

Step 4: Choose an integer between 1 and $(n-1)$. This will be the number used to create the cypher. Label this number s.

Step 5: Write a message using your character set.

Step 6: Replace every character in your message by the integer with which it was paired in Step 3.

Step 7: For every number, x, from Step 7, perform the addition $x + s \pmod{n}$.

Step 8: Replace every number found in Step 7 with the character with which it was paired in Step 3. This is your cyphertext.

To decrypt your cyphertext, reverse the steps above.

Step 1: Replace the cyphertext characters with the paired values.

Step 2: For each value x, perform the *subtraction* $x - s \pmod{n}$.

Step 3: Replace the numbers from Step 2 with their paired characters from the character set.

The message is then deciphered.

Design Your Own Cypher Using Multiplication

A cypher is a message written in such a way as to mask its contents. Changing a message into its cypher form is called encryption. Decryption or deciphering is the process of changing a cyphertext message back into the original (legible) message. One process of encryption is to scramble the letters, symbols, and punctuation of a message according to a mathematical rule. One rule that could be used for such a cypher is multiplication in a chosen modulus. In this project, you will create such a cypher, encrypt a message, then decrypt the message.

Step 1: Choose the letters, symbols, and punctuation marks you want to allow in your messages. This should include at least the uppercase letters and a space character. This is your character set.

Step 2: Count the number of characters you will use. Label this number n.

Step 3: Pair each character of your character set an integer from 0 to $(n - 1)$. Do not assign more than one character to an integer.

Step 4: Choose an integer, labeled s, between 1 and $(n - 1)$ so that GCF $(n, s) = 1$. This will be the number used to create the cypher.

Step 5: Write a message using your character set.

Step 6: Replace every character in your message by the integer with which it was paired in Step 3.

Step 7: For every number, x, from Step 6, perform the multiplication $x - s \pmod{n}$.

Step 8: Replace every number found in Step 7 with the character with which it was paired in Step 3. This is your cyphertext.

Before beginning to decrypt in this cypher, you need to know the **multiplicative inverse** of the value you chose as s.

Step 1: The multiplicative inverse of s is the number that, when multiplied by s in your modulus, equals 1. To find this, you will have to multiply s and every number between 2 and $(n - 1)$ until the product is 1 (mod n). Once this number is found, the message can be decrypted. Call this number r.

Step 2: To decrypt your cyphertext, replace the cyphertext characters with the paired values.

Step 3: For each of the value, x, perform the multiplication $x \times r \pmod{n}$.

Step 4: Replace the numbers from Step 3 with their paired characters from the character set.

The message is then deciphered.

Key Terms

- common ratio
- geometric sequence

Key Concepts

- Geometric sequence.
- Finding an arbitrary term in a geometric sequence.
- Constant ratio.
- Finding the sum of a finite geometric sequence.
- Applying arithmetic sequences.

Formulas

$$a_n = a_1 r^{n-1}$$

$$s_n = a_1\left(\frac{1 - r^{n-1}}{1 - r}\right)$$