Chapter 2: Logic

Figure 1: Logic is key to a well-reasoned argument, in both math and law.

What is logic? **Logic** is the study of reasoning, and it has applications in many fields, including philosophy, law, psychology, digital electronics, and computer science.

In law, constructing a well-reasoned, logical argument is extremely important. The main goal of arguments made by lawyers is to convince a judge and jury that their arguments are valid and well-supported by the facts of the case, so the case should be ruled in their favor. Think about Thurgood Marshall arguing for desegregation in front of the U.S. Supreme Court during the *Brown v. Board of Education of Topeka* lawsuit in 1954, or Ruth Bader Ginsburg arguing for equality in social security benefits for both men and women under the law during the mid-1970s. Both these great minds were known for the preparation and thoroughness of their logical legal arguments, which resulted in victories that advanced the causes they fought for. Thurgood Marshall and Ruth Bader Ginsburg would later become well respected justices on the U.S. Supreme Court themselves.

In this chapter, we will explore how to construct well-reasoned logical arguments using varying structures. Your ability to form and comprehend logical arguments is a valuable tool in many areas of life, whether you're planning a dinner date, negotiating the purchase of a new car, or persuading your boss that you deserve a raise.

2.1 Statements and Quantifiers

Figure 2: Construction of a logical argument, like that of a house, requires you to begin with the right parts.

Learning Objectives

After completing this section, you should be able to:

1. Identify logical statements.
2. Represent statements in symbolic form.
3. Negate statements in words.
4. Negate statements symbolically.
5. Translate negations between words and symbols.
6. Express statements with quantifiers of all, some, and none.
7. Negate statements containing quantifiers of all, some, and none.

Have you ever built a club house, tree house, or fort with your friends? If so, you and your friends likely started by gathering some tools and supplies to work with, such as hammers, saws, screwdrivers, wood, nails, and screws. Hopefully, at least one member of your group had some knowledge of how to use the tools correctly and helped to direct the construction project. After all, if your house isn't built on a strong foundation, it will be weak and could possibly fall apart during the next big storm. This same foundation is important in logic.

In this section, we will begin with the parts that make up all logical arguments. The building block of any logical argument is a **logical statement**, or simply a statement. A logical statement has the form of a complete sentence, and it must make a claim that can be identified as being true or false.

When making arguments, sometimes people make false claims. When evaluating the strength or validity of a logical argument, you must also consider the **truth values**, or the identification of true or false, of all the statements used to support the argument. While a false statement is still considered a logical statement, a strong logical argument starts with true statements.

Identifying Logical Statements

Figure 3: Not all roses are red.

An example of logical statement with a false truth value is, "All roses are red." It is a logical statement because it has the form of a complete sentence and makes a claim that can be determined to be either true or false. It is a false statement because not all roses are red: some roses are red, but there are also roses that are pink, yellow, and white. Requests, questions, or directives may be complete sentences, but they are not logical statements because they cannot be determined to be true or false. For example, suppose someone said to you, "Please, sit down over there." This request does not make a claim and it cannot be identified as true or false; therefore, it is not a logical statement.

Example 1 **Identifying Logical Statements**

Determine whether each of the following sentences represents a logical statement. If it is a logical statement, determine whether it is true or false.

1. Tiger Woods won the Master's golf championship at least five times.
2. Please, sit down over there.
3. All cats dislike dogs.

Solution

1. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. As of 2021, this statement is true: Tiger Woods won the Master's in 1997, 2001, 2002, 2005 and 2019.
2. This is not a logical statement because, although it is a complete sentence, this request does not make a claim that can be identified as being either true or false.

3. This is a logical statement because it is a complete sentence that makes a claim that can be identified as being true or false. This statement is false because some cats do like some dogs.

Representing Statements in Symbolic Form

When analyzing logical arguments that are made of multiple logical statements, **symbolic form** is used to reduce the amount of writing involved. Symbolic form also helps visualize the relationship between the statements in a more concise way in order to determine the strength or validity of an argument. Each logical statement is represented symbolically as a single lowercase letter, usually starting with the letter p.

To begin, you will practice how to write a single logical statement in symbolic form. This skill will become more useful as you work with compound statements in later sections.

Example 2 **Representing Statements Using Symbolic Form**

Write each of the following logical statements in symbolic form.

1. Barry Bonds holds the Major League Baseball record for total career home runs.
2. Some mammals live in the ocean.
3. Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020.

Solution

1. p: Barry Bonds holds the Major League Baseball record for total career home runs. The statement is labeled with a p . Once the statement is labeled, use p as a replacement for the full written statement to write and analyze the argument symbolically.
2. q: Some mammals live in the ocean. The letter q is used to distinguish this statement from the statement in question 1, but any lower-case letter may be used.
3. r: Ruth Bader Ginsburg served on the U.S. Supreme Court from 1993 to 2020. When multiple statements are present in later sections, you will want to be sure to use a different letter for each separate logical statement.

WHO KNEW?

Mathematics is not the only language to use symbols to represent thoughts or ideas. The Chinese and Japanese languages use symbols known as Hanzi and Kanji, respectively, to represent words and phrases. At one point, the American musician Prince famously changed his name to a symbol representing love.

BBC Prince Symbol Article

Negating Statements

Consider the false statement introduced earlier, “All roses are red.” If someone said to you, “All roses are red,” you might respond with, “Some roses are not red.” You could then strengthen your argument by providing additional statements, such as, "There are also white roses, yellow roses, and pink roses, to name a few."

The **negation of a logical statement** has the opposite truth value of the original statement. If the original statement is false, its negation is true, and if the original statement is true, its negation is false. Most logical statements can be negated by simply adding or removing the word *not*. For example, consider the statement, "Emma Stone has green eyes." The negation of this statement would be, "Emma Stone does not have green eyes." The table below gives some other examples.

Logical Statement	Negation
Gordon Ramsey is a chef.	Gordon Ramsey is not a chef.
Tony the Tiger does not have spots.	Tony the Tiger has spots.

The way you phrase your argument can impact its success. If someone presents you with a false statement, the ability to rebut that statement with its negation will provide you with the tools necessary to emphasize the correctness of your position.

Example 3 **Negating Logical Statements**

Write the negation of each logical statement in words.

1. Michael Phelps was an Olympic swimmer.
2. Tom is a cat.
3. Jerry is not a mouse.

Solution

1. Add the word *not* to negate the affirmative statement: "Michael Phelps was not an Olympic swimmer."
2. Add the word *not* to negate the affirmative statement: "Tom is not a cat."
3. Remove the word *not* to negate the negative statement: "Jerry is a mouse."

Negating Logical Statements Symbolically

The symbol for negation, or not, in logic is the tilde, . So, not p is represented as $\sim p$. To negate a statement symbolically, remove or add a tilde. The negation of not (not p) is p. Symbolically, this equation is $\sim(\sim p) = p$.

Example 4 **Negating Logical Statements Symbolically**

Write the negation of each logical statement symbolically.

1. p: Michael Phelps was an Olympic swimmer.
2. r: Tom is not a cat.
3. $\sim q$: Jerry is not a mouse.

Solution

1. To negate an affirmative logical statement symbolically, add a tilde: $\sim p$.
2. Because the symbol for this statement is r, its negation is $\sim r$.
3. The symbol for this statement is $\sim q$, so to negate it we simply remove the , because $\sim(\sim q) = q$. The answer is q.

Translating Negations Between Words and Symbols

In order to analyze logical arguments, it is important to be able to translate between the symbolic and written forms of logical statements.

Example 5 **Translating Negations Between Words and Symbols**

Given the statements:

r: Elmo is a red Muppet.

p: Ketchup is not a vegetable.

1. Write the symbolic form of the statement, "Elmo is not a red Muppet."
2. Translate the statement $\sim p$ into words.

Solution

1. "Elmo is not a red muppet" is the negation of "Elmo is a red muppet," which is represented as r. Thus, we would write "Elmo is not a red muppet" symbolically as $\sim r$.
2. Because p is the symbol representing the statement, "Ketchup is not a vegetable," $\sim p$ is equivalent to the statement, "Ketchup is a vegetable."

Expressing Statements with Quantifiers of All, Some, or None

A **quantifier** is a term that expresses a numerical relationship between two sets or categories. For example, all squares are also rectangles, but only some rectangles are squares, and no squares are circles. In this example, *all, some,* and *none* are quantifiers. In a logical argument, the logical statements made to support the argument are called **premises**, and the judgment made based on the premises is called the **conclusion**. Logical arguments that begin with specific premises and attempt to draw more general conclusions are called **inductive arguments**.

Consider, for example, a parent walking with their three-year-old child. The child sees a cardinal fly by and points it out. As they continue on their walk, the child notices a robin land on top of a tree and a duck flying across to land on a pond. The child recognizes that cardinals, robins, and ducks are all birds, then excitedly declares, "All birds fly!" The child has just made an inductive argument. They noticed that three different specific types of birds all fly, then synthesized this information to draw the more general conclusion that all birds can fly. In this case, the child's conclusion is false.

The specific premises of the child's argument can be paraphrased by the following statements:

- Premise: Cardinals are birds that fly.
- Premise: Robins are birds that fly.
- Premise: Ducks are birds that fly.

The general conclusion is: "All birds fly!"

All inductive arguments should include at least three specific premises to establish a pattern that supports the general conclusion. To counter the conclusion of an inductive argument, it is necessary to provide a counter example. The parent can tell the child about penguins or emus to explain why that conclusion is false.

On the other hand, it is usually impossible to prove that an inductive argument is true. So, inductive arguments are considered either strong or weak. Deciding whether an inductive argument is strong or weak is highly subjective and often determined by the background knowledge of the person making the judgment. Most hypotheses put forth by scientists using what is called the "scientific method" to conduct experiments are based on inductive reasoning.

In the following example, we will use quantifiers to express the conclusion of a few inductive arguments.

Example 6 **Drawing General Conclusions to Inductive Arguments Using Quantifiers**

For each series of premises, draw a logical conclusion to the argument that includes one of the following quantifiers: all, some, or none.

1. Squares and rectangles have four sides. A square is a parallelogram, and a rectangle is a parallelogram. What conclusion can be drawn from these premises?
2. $1 + 2 = 3$, $6 + 7 = 13$, and $23 + 24 = 47$. Of these, 1 and 2, 6 and 7, and 23 and 24 are consecutive integers; 3, 13, and 47 are odd numbers. What conclusion can be drawn from these premises?
3. Sea urchins live in the ocean, and they do not breathe air. Sharks live in the ocean, and they do not breathe air. Eels live in the ocean, and they do not breath air. What conclusion can be drawn from these premises?

Solution

1. The conclusion you would likely come to here is "Some four-sided figures are parallelograms." However, it would be incorrect to say that all four-sided figures are parallelograms because there are some four-sided figures, such as trapezoids, that are not parallelograms. This is a false conclusion.
2. From these premises, you may draw the conclusion "All sums of two consecutive counting numbers result in an odd number." Most inductive arguments cannot be proven true, but several mathematical properties can be. If we let n represent our first counting number, then $n + 1$ would be the next counting number and $n + (n + 1) = 2n + 1$. Because $2n$ is divisible by two, it is an even number, and if you add

one to any even number the result is always an odd number. Thus, the conclusion is true!

3. Based on the premises provided, with no additional knowledge about whales or dolphins, you might conclude "No creatures that live in the ocean breathe air." Even though this conclusion is false, it still follows from the known premises and thus is a logical conclusion based on the evidence presented. Alternatively, you could conclude "Some creatures that live in the ocean do not breathe air." The quantifier you choose to write your conclusion with may vary from another person's based on how persuasive the argument is. There may be multiple acceptable answers.

CHECKPOINT

It is not possible to prove definitively that an inductive argument is true or false in most cases.

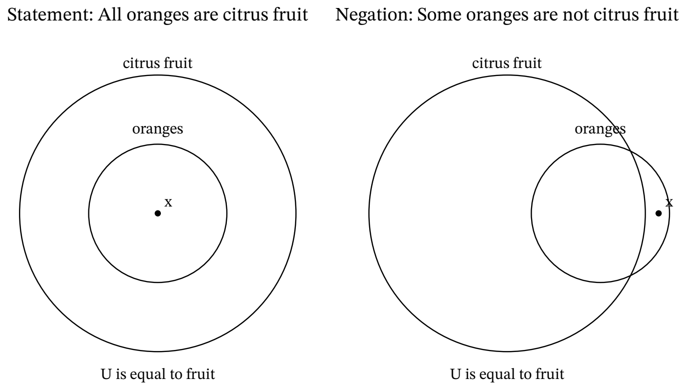
Negating Statements Containing Quantifiers

Recall that the negation of a statement will have the opposite truth value of the original statement. There are four basic forms that logical statements with quantifiers take on.

- All A are B.
- Some A are B.
- No A are B.
- Some A are not B.

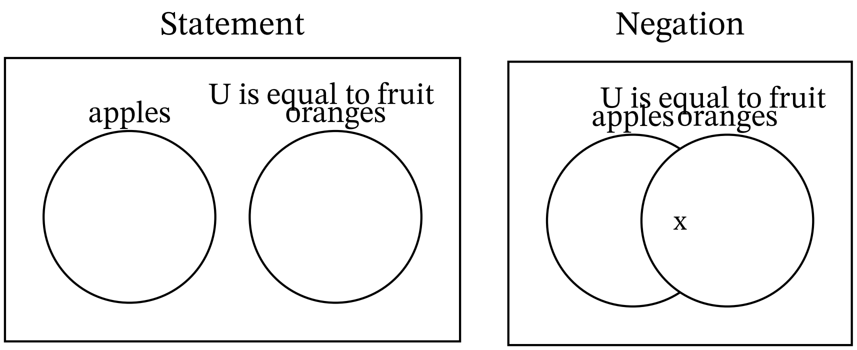
The negation of logical statements that use the quantifiers *all*, *some*, or *none* is a little more complicated than just adding or removing the word *not*.

For example, consider the logical statement, "All oranges are citrus fruits." This statement expresses as a subset relationship. The set of oranges is a subset of the set of citrus fruit. This means that there are no oranges that are outside the set of citrus fruit. The negation of this statement would have to break the subset relationship. To do this, you could say, "At least one orange is not a citrus fruit." Or, more concisely, "Some oranges are not citrus fruit." It is tempting to say "No oranges are citrus fruit," but that would be incorrect. Such a statement would go beyond breaking the subset relationship, to stating that the two sets have nothing in common. The negation of "A is a subset of B" would be to state that "A is not a subset of B," as depicted by the Venn diagram in .



The statement, “All oranges are citrus fruit,” is true, so its negation, “Some oranges are not citrus fruit,” is false.

Now, consider the statement, “No apples are oranges.” This statement indicates that the set of apples is disjointed from the set of oranges. The negation must state that the two are not disjoint sets, or that the two sets have a least one member in common. Their intersection is not empty. The negation of the statement, “*A* intersection *B* is the empty set,” is the statement that “ *A* intersection *B* is not empty,” as depicted in the Venn diagram in .



The negation of the true statement “No apples are oranges,” is the false statement, “Some apples are oranges.”

summarizes the four different forms of logical statements involving quantifiers and the forms of their associated negations, as well as the meanings of the relationships between the two categories or sets *A* and *B*.

Logical Statements with Quantifiers	Negation of Logical Statements w/Quantifiers
Form: All A are B. Means: A is a subset of B, $A \subset B$. All zebras have stripes. (True)	Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some zebras do not have stripes. (False)
Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \varnothing$. Some fish are sharks. (True)	Form: No A are B. Means: A intersection B is empty, $A \cap B = \varnothing$. No fish are sharks. (False)
Form: No A are B. Means: A intersection B is empty, $A \cap B = \varnothing$. No trees are evergreens. (False)	Form: Some A are B. Means: A intersection B is not empty, $A \cap B \neq \varnothing$. Some trees are evergreens. (True)
Form: Some A are not B. Means: A is not a subset of B, $A \not\subset B$. Some horses are not mustangs. (True)	Form: All A are B. Means: A is a subset of B, $A \subset B$. All horses are mustangs. (False)

We covered sets in great detail in Chapter 1. To review, " A is a subset of B" means that every member of set A is also a member of set B. The intersection of two sets A and B is the set of all elements that they share in common. If A intersection B is the empty set, then sets A and B do not have any elements in common. The two sets do not overlap. They are disjoint.

VIDEO

Logic Part 1A: Logic Statements and Quantifiers

Example 7 **Negating Statements Containing Quantifiers *All*, *Some*, or *None***

Given the statements:

p: All leopards have spots.
r: Some apples are red.
s: No lemons are sweet.

Write each of the following symbolic statements in words.

1. $\sim p$
2. r
3. $\sim s$

Solution

1. The statement "All leopards have spots" is p and has the form "All A are B." The negation will have the form "Some A are not B." The negation of p is the statement, "Some leopards do not have spots."

2. The statement “Some apples are red” has the form “Some A are B.” This indicates that the categories A and B overlap or intersect. The negation will have the form, “No A are B,” indicating that A and B do not intersect. This results in the opposite truth value of the original statement, so the negation of “Some apples are red” is the statement: “No apples are red.”
3. Because s is the statement: “No lemons are sweet,” s is asserting that the set of lemons does not intersect with the set of sweet things. The negation of s, $\sim s$, must make the opposite claim. It must indicate that the set of lemons intersects with the set of sweet things. This means at least one lemon must be sweet. The statement, “Some lemons are sweet” is $\sim s$. The negation of the statement, “No A are B,” is the statement, “Some A are B,” as indicated in .

Key Terms

- logic
- logical statement
- truth values
- symbolic form
- negation of a logical statement
- quantifier
- premises
- conclusion
- inductive logical arguments

Key Concepts

- Logical statements have the form of a complete sentence and make claims that can be identified as true or false.
- Logical statements are represented symbolically using a lowercase letter.
- The negation of a logical statement has the opposite truth value of the original statement.
- Be able to
  - Determine whether a sentence represents a logical statement.
- Write and translate logical statements between words and symbols.
- Negate logical statements, including logical statements containing quantifiers of *all, some, and none*.

video

- Logic Part 1A: Logic Statements and Quantifiers

2.2 Compound Statements

Figure 6: A person seeking their driver's license must pass two tests. A compound statement can be used to explain performance on both tests at once.

Learning Objectives

After completing this section, you should be able to:

1. Translate compound statements into symbolic form.
2. Translate compound statements in symbolic form with parentheses into words.
3. Apply the dominance of connectives.

Suppose your friend is trying to get a license to drive. In most places, they will need to pass some form of written test proving their knowledge of the laws and rules for driving safely. After passing the written test, your friend must also pass a road test to demonstrate that they can perform the physical task of driving safely within the rules of the law.

Consider the statement: "My friend passed the written test, but they did not pass the road test." This is an example of a **compound statement**, a statement formed by using a **connective** to join two independent clauses or logical statements. The statement, "My friend passed the written test," is an independent clause because it is a complete thought or idea that can stand on its own. The second independent clause in this compound statement is, "My friend did not pass the road test." The word "but" is the connective used to join these two statements together, forming a compound statement. So, did your friend acquire their driving license?.

This section introduces common logical connectives and their symbols, and allows you to practice translating compound statements between words and symbols. It also explores the order of operations, or dominance of connectives, when a single compound statement uses multiple connectives.

Common Logical Connectives

Understanding the following logical connectives, along with their properties, symbols, and names, will be key to applying the topics presented in this chapter. The chapter will discuss each connective introduced here in more detail.

The joining of two logical statements with the word "and" or "but" forms a compound statement called a **conjunction**. In logic, for a conjunction to be true, all the independent logical statements that make it up must be true. The symbol for a conjunction is $\wedge$. Consider the compound statement, "Derek Jeter played professional baseball for the New York Yankees, and he was a shortstop." If p represents the statement, "Derrick Jeter played professional baseball for the New York Yankees," and if q represents the statement, "Derrick Jeter was a short stop," then the conjunction will be written symbolically as $p \wedge q$.

The joining of two logical statements with the word "or" forms a compound statement called a **disjunction**. Unless otherwise specified, a disjunction is an inclusive *or* statement, which means the compound statement formed by joining two independent clauses with the word *or* will be true if a least one of the clauses is true. Consider the compound statement, "The office manager ordered cake for for an employee's birthday or they ordered ice cream." This is a disjunction because it combines the independent clause, "The office manager ordered cake for an employee's birthday," with the independent clause, "The office manager ordered ice cream," using the connective, *or*. This disjunction is true if the office manager ordered only cake, only ice cream, or they ordered both cake and ice cream. Inclusive *or* means you can have one, or the other, or both!

Joining two logical statements with the word *implies*, or using the phrase "if *first statement*, then*second statement,"* is called a **conditional** or *implication*. The clause associated with the "if" statement is also called the **hypothesis** or *antecedent*, while the clause following the "then" statement or the word *implies* is called the **conclusion** or *consequent*. The conditional statement is like a one-way contract or promise. The only time the conditional statement is false, is if the hypothesis is true and the conclusion is false. Consider the following conditional statement, "If Pedro does his homework, then he can play video games." The hypothesis/antecedent is the statement following the word *if*, which is "Pedro does/did his homework." The conclusion/consequent is the statement following the word *then*, which is "Pedro can play his video games."

Joining two logical statements with the connective phrase "if and only if" is called a **biconditional**. The connective phrase "if and only if" is represented by the symbol, $\leftrightarrow$. In the biconditional statement, $p \leftrightarrow q$, p is called the hypothesis or antecedent and q is called the conclusion or consequent. For a biconditional statement to be true, the truth values of p and q must match. They must both be true, or both be false.

The table below lists the most common connectives used in logic, along with their symbolic forms, and the common names used to describe each connective.

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not		negation
if …, then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

CHECKPOINT

These connectives, along with their names, symbols, and properties, will be used throughout this chapter and should be memorized.

Example 1 **Associate Connectives with Symbols and Names**

For each of the following connectives, write its name and associated symbol.

1. or
2. implies
3. but

Solution

1. A compound statement formed with the connective word *or* is called a disjunction, and it is represented by the $\vee$ symbol.
2. A compound statement formed with the connective word *implies* or phrase "*if …, then*" is called a conditional statement or implication and is represented by the $\rightarrow$ symbol.
3. A compound statement formed with the connective words *but* or *and* is called a conjunction, and it is represented by the $\wedge$ symbol.

Translating Compound Statements to Symbolic Form

To translate a compound statement into symbolic form, we take the following steps.

1. Identify and label all independent affirmative logical statements with a lower case letter, such as p, q, or r.
2. Identify and label any negative logical statements with a lowercase letter preceded by the negation symbol, such as $\sim p$, $\sim q$, or $\sim r$.
3. Replace the connective words with the symbols that represent them, such as $\wedge, \vee, \rightarrow$, or $\leftrightarrow$.

   Consider the previous example of your friend trying to get their driver's license. Your friend passed the written test, but they did not pass the road test. Let p represent the statement, "My friend passed the written test." And, let $\sim q$ represent the statement, "My friend did not pass the road test." Because the connective *but* is logically equivalent to the word *and*, the symbol for *but* is the same as the symbol for *and* replace *but* with the symbol $\wedge$. The compound statement is symbolically written as: $p \wedge \sim q$. My friend passed the written test, but they did not pass the road test.

CHECKPOINT

When translating English statements into symbolic form, do not assume that every connective "and" means you are dealing with a compound statement. The statement, "The stripes on the U.S. flag are red and white," is a simple statement. The word "white" doesn't express a complete thought, so it is not an independent clause and does not get its own symbol. This statement should be represented by a single letter, such as s: The stripes on the U.S. flag are red and white.

Example 2 **Translating Compound Statements into Symbolic Form**

Let p represent the statement, "It is a warm sunny day," and let q represent the statement, "the family will go to the beach." Write the symbolic form of each of the following compound statements.

1. If it is a warm sunny day, then the family will go to the beach.
2. The family will go to the beach, and it is a warm sunny day.
3. The family will not go to the beach if and only if it is not a warm sunny day.
4. The family not go to the beach, or it is a warm sunny day.

Solution

1. Replace "it is a warm sunny day" with p. Replace "the family will go to the beach." with q. Next. Next, because the connective is *if …, then* place the conditional symbol, $\rightarrow$, between p and q. The compound statement is written symbolically as: $p \rightarrow q$.
2. Replace "The family will go to the beach" with q. Replace "it is a warm sunny day." with p. Next, because the connective is *and*, place the $\wedge$ symbol between q and p. The compound statement is written symbolically as: $q \wedge p$.
3. Replace "The family will not go to the beach. with $\sim q$. Replace "it is not a warm sunny day" with $\sim p$. Next, because the connective is *or*, *if and only if*, place the biconditional symbol, $\leftrightarrow$ between $\sim q$ and $\sim p$. The compound statement is written symbolically as: $\sim q \leftrightarrow \sim p$.
4. Replace "The family will not go to the beach" with $\sim q$. Replace "it is a warm sunny day" with p. Next, because the connective is *or*, place the $\vee$ symbol between $\sim q$ and p. The compound statement is written symbolically as: $\sim q \vee p$.

Translating Compound Statements in Symbolic Form with Parentheses into Words

When parentheses are written in a logical argument, they group a compound statement together just like when calculating numerical or algebraic expressions. Any statement in parentheses should be treated as a single component of the expression. If multiple parentheses are present, work with the inner most parentheses first.

Consider your friend's struggles to get their license to drive. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." The statement $(p \wedge q) \rightarrow r$ can be translated into words as follows: the statement $p \wedge q$ is grouped together to form the hypothesis of the conditional statement and r is the conclusion. The conditional statement has the form "*if* $p \wedge q$, *then* r." Therefore, the written form of this statement is: "If my friend passed the written test and they passed the road test, then my friend received a driver's license."

Sometimes a compound statement within parentheses may need to be negated as a group. To accomplish this, add the phrase, "it is not the case that" before the translation of the phrase in parentheses. For example, using p, q, and r of your friend obtaining a license, let's translate the statement $\sim(p \wedge q) \rightarrow \sim r$ into words.

In this case, the hypothesis of the conditional statement is $\sim(p \wedge q)$ and the conclusion is $\sim r$. To negate the hypothesis, add the phrase "it is not the case" before translating what is in parentheses. The translation of the hypothesis is the sentence, "It is not the case that my friend passed the written test and they passed the road test," and the translation of the conclusion is, "My friend did not receive a driver's license." So, a translation of the complete conditional statement, $\sim(p \wedge q) \rightarrow \sim r$ is: "If it is not the case that my friend passed the written test and the road test, then my friend did not receive a driver's license."

CHECKPOINT

It is acceptable to interchange proper names with pronouns and remove repeated phrases to make the written statement more readable, as long the meaning of the logical statement is not changed.

VIDEO

Logic Part 1B: Compound Statements, Connectives and Symbols

Example 3 **Translating Compound Statements in Symbolic Form with Parentheses into Words**

Let p represent the statement, "My child finished their homework," let q represent the statement, "My child cleaned her room," let r represent the statement, "My child played video games," and let s represent the statement, "My child streamed a movie." Translate each of the following symbolic statements into words.

1. $\sim(p \wedge q)$
2. $(p \wedge q) \rightarrow (r \vee s)$

3. $\sim(r \vee s) \leftrightarrow \sim(p \wedge q)$

Solution

1. Replace with "It is not the case," and $\wedge$ with "and." One possible translation is: "It is not the case that my child finished their homework and cleaned their room."
2. The hypothesis of the conditional statement is, "My child finished their homework and cleaned their room." The conclusion of the conditional statement is, "My child played video games or streamed a movie." One possible translation of the entire statement is: "If my child finished their homework and cleaned their room, then they played video games or streamed a movie."
3. The hypothesis of the biconditional statement is $\sim(r \vee s)$ and is written in words as: "It is not the case that my child played video games or streamed a movie." The conclusion of the biconditional statement is $\sim(p \wedge q)$, which translates to: "It is not the case that my child finished their homework and cleaned their room." Because the biconditional, $\leftrightarrow$ translates to *if and only if*, one possible translation of the statement is: "It is not the case that my child played video games or streamed a movie if and only if it is not the case that my child finished their homework and cleaned their room."

The Dominance of Connectives

The order of operations for working with algebraic and arithmetic expressions provides a set of rules that allow consistent results. For example, if you were presented with the problem $1 + 3 \times 2$, and you were not familiar with the order of operation, you might assume that you calculate the problem from left to right. If you did so, you would add 1 and 3 to get 4, and then multiply this answer by 2 to get 8, resulting in an incorrect answer. Try inputting this expression into a scientific calculator. If you do, the calculator should return a value of 7, not 8.

Scientific Calculator

The order of operations for algebraic and arithmetic operations states that all multiplication must be applied prior to any addition. Parentheses are used to indicate which operation—addition or multiplication—should be done first. Adding parentheses can change and/or clarify the order. The parentheses in the expression $1 + (3 \times 2)$ indicate that 3 should be multiplied by 2 to get 6, and then 1 should be added to 6 to get 7: $1 + (3 \times 2) = 7$.

As with algebraic expressions, there is a set of rules that must be applied to compound logical statements in order to evaluate them with consistent results. This set of rules is called the **dominance of connectives**. When evaluating compound logical statements, connectives are evaluated from least dominant to most dominant as follows:

1. Parentheses are the least dominant connective. So, any expression inside parentheses must be evaluated first. Add as many parentheses as needed to any statement to specify the order to evaluate each connective.
2. Next, we evaluate negations.

3. Then, we evaluate conjunctions and disjunctions from left to right, because they have equal dominance.
4. After evaluating all conjunctions and disjunctions, we evaluate conditionals.
5. Lastly, we evaluate the most dominant connective, the biconditional. If a statement includes multiple connectives of equal dominance, then we will evaluate them from left to right.

   See for a visual breakdown of the dominance of connectives.

Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
	Negation	~	
	Disjunction/Conjunction	∨, ∧	Left to right or add parentheses to specify order because or/and have equal dominance.
	Conditional	→	
Most Dominant	Biconditional	↔	Last

Let's revisit your friend's struggles to get their driver's license. Let p represent the statement, "My friend passed the written test," let q represent the statement, "My friend passed the road test," and let r represent the statement, "My friend received a driver's license." Let's use the dominance of connectives to determine how the compound statement $p \wedge \sim q \rightarrow r$ should be evaluated.

Step 1: There are no parentheses, which is least dominant of all connectives, so we can skip over that.

Step 2: Because negation is the next least dominant, we should evaluate $\sim q$ first. We could add parentheses to the statement to make it clearer which connecting needs to be evaluated first: $p \wedge \sim q \rightarrow r$ is equivalent to $p \wedge (\sim q) \rightarrow r$.

Step 3: Next, we evaluate the conjunction, $\wedge$. $p \wedge (\sim q) \rightarrow r$ is equivalent to $(p \wedge (\sim q)) \rightarrow r$.

Step 4: Finally, we evaluate the conditional, $\rightarrow$, as this is the most dominant connective in this compound statement.

WHO KNEW?

When using spreadsheet applications, like Microsoft Excel or Google Sheets, have you noticed that core functions, such as sum, average, or rate, have the same name and syntax for use? This is not a coincidence. Various standards organizations have developed requirements that software developers must implement to meet the conditions set by governments and large customers worldwide. The OpenDocument Format OASIS Standard enables transferring data between different office productivity applications and was approved by the International Standards Organization (ISO) and IEC on May 6, 2006.

Prior to adopting these standards, a government entity, business, or individual could lose access to their own data simply because it was saved in a format no longer supported by a proprietary software product—even data they were required by law to preserve, or data they simply wished to maintain access to.

Just as rules for applying the dominance of connectives help maintain understanding and consistency when writing and interpreting compound logical statements and arguments, the standards adopted for database software maintain global integrity and accessibility across platforms and user bases.

Example 4 **Applying the Dominance of Connectives**

For each of the following compound logical statements, add parentheses to indicate the order to evaluate the statement. Recall that parentheses are evaluated innermost first.

1. $p \wedge \sim q \vee r$
2. $q \rightarrow\sim p \wedge r$
3. $\sim (p \vee q) \leftrightarrow\sim p \wedge \sim q$

Solution

1. Because negation is the least dominant connective, we evaluate it first: $p \wedge (\sim q) \vee r$. Because conjunction and disjunction have the same dominance, we evaluate them left to right. So, we evaluate the conjunction next, as indicated by the additional set of parentheses: $(p \wedge (\sim q)) \vee r$. The only remaining connective is the disjunction, so it is evaluated last, as indicated by the third set of parentheses. The complete solution is: $((p \wedge (\sim q)) \vee r)$.
2. Negation has the lowest dominance, so it is evaluated first: $q \rightarrow (\sim p) \wedge r$. The remaining connectives are the conditional and the conjunction. Because conjunction has a lower precedence than the conditional, it is evaluated next, as indicated by the second set of parentheses: $q \rightarrow ((\sim p) \wedge r)$. The last step is to evaluate the conditional, as indicated by the third set of parentheses: $(q \rightarrow ((\sim p) \wedge r))$.
3. This statement is known as De Morgan's Law for the negation of a disjunction. It is always true. Section 2.6 of this chapter will explore De Morgan's Laws in more detail.
   - First, we evaluate the negations on the right side of the biconditional prior to the conjunction.

- Then, we evaluate the disjunction on the left side of the biconditional, followed by the negation of the disjunction on the left side.
- Lastly, after completely evaluating each side of the biconditional, we evaluate the biconditional. It does not matter which side you begin with.

  The final solution is: $(\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$.

WORK IT OUT

Logic Terms and Properties – Matching Game

> Materials: For every group of four students, include 30 cards (game, trading, or playing cards), 30 individual clear plastic gaming card sleeves, and 30 card-size pieces of paper. Prepare a list of 60 questions and answers ahead of time related to definitions and problems in Statements and Quantifiers and Compound Statements. Provide each group of four students with 20 questions and their associated answers. Instruct each group to select 15 of the 20 questions, and then, for each problem selected, create one card with the question and one card with the answer. Instruct the groups to then place each problem and answer in a separate card sleeve. Once they create 15 problem cards and 15 answer cards, have students shuffle the set of cards.
>
> To play the game, groups should exchange their card decks to ensure no team begins playing with the deck that they created. Split each four-person group into teams of two students. After shuffling the cards, one team lays cards face down on their desk in a five-by-six grid. The other team will go first.
>
> Each player will have a turn to try matching the question with the correct answer by flipping two cards to the face up position. If a team makes a match, they get to flip another two cards; if they do not get a match, they flip the cards face down and it is the other team's turn to flip over two cards. The game continues in this manner until teams match all question cards with their corresponding answer cards. The team with the most set of matching cards wins.

In the first module of this chapter, we evaluated the truth value of negations. In the following modules, we will discuss how to evaluate conjunctions, disjunctions, conditionals, and biconditionals, and then evaluate compound logical statements using truth tables and our knowledge of the dominance of connectives.

Key Terms

- compound statement
- connective
- conjunction
- disjunction
- conditional
- hypothesis
- conclusion
- biconditional
- dominance of connectives

Key Concepts

- Logical connectives are used to form compound logical statements by using words such as *and, or, and if …, then*.

- A conjunction is a compound logical statement formed by combining two statements with the words "and" or "but." If the two independent clauses are represented by p and q, respectively, then the conjunction is written symbolically as $p \wedge q$. For the conjunction to be true, both p and q must be true.
- A disjunction joins two logical statements with the *or* connective. In, logic *or* is inclusive. For an *or* statement to be true at least one statement must be true, but both may also be true.
- A conditional statement has the form if p, then q, where p and q are logical statements. The only time the conditional statement is false is when p is true, and q is false.
- The biconditional statement is formed using the connective if and only if for the biconditional statement to be true, the true values of p and q, must match. If p is true then q must be true, if p is false, then q must be false.
- Translate compound statements between words and symbolic form.

Connective	Symbol	Name
and but	$\wedge$	conjunction
or	$\vee$	disjunction, inclusive or
not		negation
if …, then implies	$\rightarrow$	conditional, implication
if and only if	$\leftrightarrow$	biconditional

- The dominance of connectives explains the order in which compound logical statements containing multiple connectives should be interpreted.
- The dominance of connectives should be applied in the following order
  - Parentheses
- Negations
- Disjunctions/Conjunctions, left to right
- Conditionals
- Biconditionals

Dominance	Connective	Symbol	Evaluate
Least Dominant	Parentheses	( )	First
↓	Negation	~	↓
	Disjunction/Conjunction	$\vee$, $\wedge$	Left to right or add parentheses to specify order because or/and have equal dominance.
	Conditional	$\rightarrow$	↓
Most Dominant	Biconditional	$\leftrightarrow$	Last

video

- Logic Part 1B: Compound Statements, Connectives and Symbols

2.3 Constructing Truth Tables

Figure 9: Just like solving a puzzle, a computer programmer must consider all potential solutions in order to account for each possible outcome.

Learning Objectives

After completing this section, you should be able to:

1. Interpret and apply negations, conjunctions, and disjunctions.
2. Construct a truth table using negations, conjunctions, and disjunctions.
3. Construct a truth table for a compound statement and interpret its validity.

Are you familiar with the Choose Your Own Adventure book series written by Edward Packard? These gamebooks allow the reader to become one of the characters and make decisions that affect what happens next, resulting in different sequences of events in the story and endings based on the choices made. Writing a computer program is a little like what it must be like to write one of these books. The programmer must consider all the possible inputs that a user can put into the program and decide what will happen in each case, then write their program to account for each of these possible outcomes.

A **truth table** is a graphical tool used to analyze all the possible truth values of the component logical statements to determine the validity of a statement or argument along with all its possible outcomes. The rows of the table correspond to each possible outcome for the given logical statement identified at the top of each column. A single logical statement p has two possible truth values, true or false. In truth tables, a capital T will represent true values, and a capital F will represent false values.

In this section, you will use the knowledge built in Statements and Quantifiers and Compound Statements to analyze arguments and determine their truth value and validity.

A logical argument is valid if its conclusion follows from its premises, regardless of whether those premises are true or false. You will then explore the truth tables for negation, conjunction, and disjunction, and use these truth tables to analyze compound logical statements containing these connectives.

Interpret and Apply Negations, Conjunctions, and Disjunctions

The negation of a statement will have the opposite truth value of the original statement. When p is true, $\sim p$ is false, and when p is false, $\sim p$ is true.

Example 1 **Finding the Truth Value of a Negation**

For each logical statement, determine the truth value of its negation.

1. p: $3 + 5 = 8$.
2. q: All horses are mustangs.
3. $\sim r$: New Delhi is not the capital of India.

Solution

1. p is true because $3 + 5$ does equal 8; therefore, the negation of p, $\sim p : 3 + 5 \neq 8$, is false.
2. q is false because there are other types of horses besides mustangs, such as Clydesdales or Arabians; therefore, the negation of q, $\sim q$, is true.
3. $\sim r$ is false because New Delhi is the capital of India; therefore, the negation of $\sim r$, r, is true.

A conjunction is a logical *and* statement. For a conjunction to be true, both statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

Example 2 **Finding the Truth Value of a Conjunction**

Given $p : 4 + 7 = 11$, $q : 11 - 3 = 7$, and $r : 7 \times 11 = 77$, determine the truth value of each conjunction.

1. $p \wedge q$
2. $\sim q \wedge r$
3. $\sim p \wedge q$

Solution

1. p is true, and q is false. Because one statement is true, and the other statement is false, this makes the complete conjunction false.
2. q is false, so $\sim q$ is true, and r is true. Therefore, both statements are true, making the complete conjunction true.

3. $\sim p$ is false, and q is false. Because both statements are false, the complete conjunction is false.

CHECKPOINT

The only time a conjunction is true is if both statements that make up the conjunction are true.

A disjunction is a logical inclusive *or* statement, which means that a disjunction is true if one or both statements that form it are true. The only way a logical inclusive *or* statement is false is if both statements that form the disjunction are false.

Example 3 **Finding the Truth Value of a Disjunction**

Given $p: 4+7=11$, $q: 11-3=7$, and $r: 7 \times 11 = 77$, determine the truth value of each disjunction.

1. $p \vee q$
2. $\sim q \vee r$
3. $\sim p \vee q$

Solution

1. p is true, and q is false. One statement is true, and one statement is false, which makes the complete disjunction true.
2. q is false, so $\sim q$ is true, and r is true. Therefore, one statement is true, and the other statement is true, which makes the complete disjunction true.
3. $\sim p$ is false, and q is false. When all of the component statements are false, the disjunction is false.

In the next example, you will apply the dominance of connectives to find the truth values of compound statements containing negations, conjunctions, and disjunctions and use a table to record the results. When constructing a truth table to analyze an argument where you can determine the truth value of each component statement, the strategy is to create a table with two rows. The first row contains the symbols representing the components that make up the compound statement. The second row contains the truth values of each component below its symbol. Include as many columns as necessary to represent the value of each compound statement. The last column includes the complete compound statement with its truth value below it.

VIDEO

Logic Part 2: Truth Values of Conjunctions: Is an “AND” statement true or false?

Logic Part 3: Truth Values of Disjunctions: Is an “OR” statement true or false?

Example 4 **Finding the Truth Value of Compound Statements**

Given $p : 4 + 7 = 11$, $q : 11 - 3 = 7$, and $r : 7 \times 11 = 77$, construct a truth table to determine the truth value of each compound statement

1. $\sim p \wedge q \vee r$
2. $\sim p \vee q \wedge r$
3. $\sim (p \wedge r) \vee q$

Solution

1. **Step 1:** The statement "$\sim p \wedge q \vee r$" contains three basic logical statements, p, q, and r, and three connectives, $\sim, \wedge, \vee$. When we place parentheses in the statement to indicate the dominance of connectives, the statement becomes $(((\sim p) \wedge q) \vee r)$.

Step 2: After we have applied the dominance of connectives, we create a two row table that includes a column for each basic statement that makes up the compound statement, and an additional column for the contents of each parentheses. Because we have three sets of parentheses, we include a column for $\sim p$, the innermost parentheses, a column for $(\sim p) \wedge q$, the next set of parentheses, and $((\sim p) \wedge q) \vee r$ in the last column for the third parentheses.
Step 3: Once the table is created, we determine the truth value of each statement starting from left to right. The truth values of p, q, and r are true, false, and true, respectively, so we place T, F, and T in the second row of the table. Because p is true, $\sim p$ is false.
Step 4: Next, evaluate $\sim p \wedge q$ from the table: $\sim p$ is false, and q is also false, so $\sim p \wedge q$ is false, because a conjunction is only true if both of the statements that make it are true. Place an F in the table below its heading.
Step 5: Finally, using the table, we understand that $(\sim p \wedge q)$ is false and r is true, so the complete statement $(\sim p \wedge q) \vee r$ is false or true, which is true (because a disjunction is true whenever at least one of the statements that make it is true). Place a T in the last column of the table. The complete statement $\sim p \wedge q \vee r$ is true.

p	q	r	$\sim p$	$(\sim p) \wedge q$	$((\sim p) \wedge q) \vee r$
T	F	T	F	F	**T**

1. **Step 1:** Applying the dominance of connectives to the original compound statement $\sim p \vee q \wedge r$, we get $(((\sim p) \vee q) \wedge r)$.
   Step 2: The table needs columns for $p, q, r, \sim p, (\sim p) \vee q$, and $((\sim p) \vee q) \wedge r$.
   Step 3: The truth values of p, q, r, and $\sim p$ are the same as in Question 1.
   Step 4: Next, $\sim p \vee q$ is false or false, which is false, so we place an F below this statement in the table. This is the only time that a disjunction is false.
   Step 5: Finally, $\sim p \vee q$ and r are the conjunction of the statements, $\sim p \vee q$ and r, and so the expressions evaluate to false and true, which is false. Recall that the only time an "and" statement is true is when both statements that form it are also true. The complete statement $\sim p \vee q \wedge r$ is false.

p	q	r	$\sim p$	$(\sim p) \vee q$	$((\sim p) \vee q) \wedge r$
T	F	T	F	F	**F**

2. **Step 1:** Applying the dominance of connectives to the original statement, we have: $((\sim (p \wedge r)) \vee q)$.
   Step 2: So, the table needs the following columns: $p, q, r, p \wedge r, \sim (p \wedge r)$, and $\sim (p \wedge r) \vee q$.
   Step 3: The truth values of p, q, and r are the same as in Questions 1 and 2.
   Step 4: From the table it can be seen that $p \wedge r$ is true and true, which is true. So the negation of p and r is false, because the negation of a statement always has the opposite truth value of the original statement.
   Step 5: Finally, $\sim (p \wedge r) \vee q$ is the disjunction of $\sim (p \wedge r)$ with q, and so we have false or false, which makes the complete statement false.

p	q	r	$p \wedge r$	$\sim (p \wedge r)$	$\sim (p \wedge r) \vee q$
T	F	T	T	F	**F**

Construct Truth Tables to Analyze All Possible Outcomes

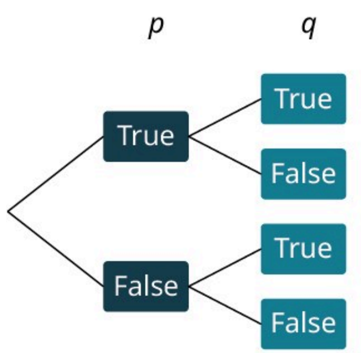
Recall from Statements and Questions that the negation of a statement will always have the opposite truth value of the original statement; if a statement p is false, then its negation $\sim p$ is true, and if a statement p is true, then its negation $\sim p$ is false. To create a truth table for the negation of statement p, add a column with a heading of $\sim p$, and for each row, input the truth value with the opposite value of the value listed in the column for p, as depicted in the table below.

Negation	
$\boldsymbol{p}$	$\boldsymbol{\sim p}$
T	F
F	T

Conjunctions and disjunctions are compound statements formed when two logical statements combine with the connectives “and” and “or” respectively. How does that change the number of possible outcomes and thus determine the number of rows in our truth table? The **multiplication principle**, also known as the *fundamental counting principle*, states that the number of ways you can select an item from a group of n items and another item from a group with m items is equal to the product of m and n. Because each proposition p and q has two possible outcomes, true or false, the multiplication principle states that there will be $2 \times 2 = 4$ possible outcomes: {TT, TF, FT, FF}.

The tree diagram and table in demonstrate the four possible outcomes for two propositions p and q.

Tree Diagram



Two Statements	
p	q
T	T
T	F
F	T
F	F

A conjunction is a logical *and* statement. For a conjunction to be true, both the statements that make up the conjunction must be true. If at least one of the statements is false, the *and* statement is false.

A disjunction is a logical inclusive *or* statement. Which means that a disjunction is true if one or both statements that make it are true. The only way a logical inclusive *or* statement is false is if both statements that make up the disjunction are false.

Conjunction (AND)		Disjunction (OR)				
p	q	$p \wedge q$		p	q	$p \vee q$
T	T	T		T	T	T
T	F	F		T	F	T
F	T	F		F	T	T
F	F	F		F	F	F

VIDEO

Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not"

Logic Part 5: What are truth tables? How do you set them up?

Example 5 **Constructing Truth Tables to Analyze Compound Statements**

Construct a truth table to analyze all possible outcomes for each of the following arguments.

1. $p \vee \sim q$
2. $\sim (p \wedge q)$
3. $(p \vee \sim q) \wedge r$

Solution

1. **Step 1:** Because there are two basic statements, p and q, and each of these has two possible outcomes, we will have 2(2) = 4 rows in our table to represent all possible outcomes: TT, TF, FT, and FF. The columns will include p, q, $\sim q$, and $p \vee \sim q$.
   Step 2: Every value in column $\sim q$ will have the opposite truth value of the corresponding value in column q: F, T, F, and T.

Step 3: To complete the last column, evaluate each element in column p with the corresponding element in column $\sim q$ using the connective *or*.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	**T**
T	F	T	**T**
F	T	F	**F**
F	F	T	**T**

1. **Step 1:** The columns will include p, q, $p \wedge q$, and $\sim (p \wedge q)$. Because there are two basic statements, p and q, the table will have four rows to account for all possible outcomes.
   Step 2: The $p \wedge q$ column will be completed by evaluating the corresponding elements in columns p and q respectively with the *and* connective.
   Step 3: The final column, $\sim (p \wedge q)$, will be the negation of the $p \wedge q$ column.

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	**F**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

2. **Step 1:** This statement has three basic statements, p, q, and r. Because each basic statement has two possible truth values, true or false, the multiplication principal indicates there are 2(2)(2) = 8 possible outcomes. So eight rows of outcomes are needed in the truth table to account for each possibility. Half of the eight possibilities must be true for the first statement, and half must be false.
   Step 2: So, the first column for statement p, will have four T's followed by four F's. In the second column for statement q, when p is true, half the outcomes for q must be true and the other half must be false, and the same pattern will repeat for when p is false. So, column q will have TT, FF, FF, FF.
   Step 3: The column for the third statement, r, must alternate between T and F. Once, the three basic propositions are listed, you will need a column for$\sim q$, $p \vee \sim q$, and $(p \vee \sim q) \wedge r$.

Step 4: The column for the negation of q, $\sim q$, will have the opposite truth value of each value in column q.
Step 5: Next, fill in the truth values for the column containing the statement $p \vee \sim q$. The *or* statement is true if at least one of p or $\sim q$ is true, otherwise it is false.
Step 6: Finally, fill in the column containing the conjunction $(p \vee \sim q) \wedge r$. To evaluate this statement, combine column $p \vee \sim q$ and column r with the *and* connective. Recall, that only time "and" is true is when both values are true, otherwise the statement is false. The complete truth table is:

p	q	r	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge r$
T	T	T	F	T	**T**
T	T	F	F	T	**F**
T	F	T	T	T	**T**
T	F	F	T	T	**F**
F	T	T	F	F	**F**
F	T	F	F	F	**F**
F	F	T	T	T	**T**
F	F	F	T	T	**F**

VIDEO
Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

Determine the Validity of a Truth Table for a Compound Statement

A logical statement is **valid** if it is always true regardless of the truth values of its component parts. To test the validity of a compound statement, construct a truth table to analyze all possible outcomes. If the last column, representing the complete statement, contains only true values, the statement is valid.

Example 6 **Determining the Validity of Compound Statements**
Construct a truth table to determine the validity of each of the following statements.

1. $\sim p \wedge q$
2. $\sim (p \wedge \sim p)$

Solution

1. **Step 1:**Because there are two statements, p and q, and each of these has two possible outcomes, there will be $2(2) = 4$ rows in our table to represent all possible outcomes: TT, TF, FT, and FF.

Step 2: The columns, will include p, q, $\sim p$ and $\sim p \wedge q$. Every value in column $\sim p$ will have the opposite truth value of the corresponding value in column p.
Step 3: To complete the last column, evaluate each element in column $\sim p$ with the corresponding element in column q using the connective *and*. The last column contains at least one false, therefore the statement $\sim p \wedge q$ is not valid.

p	q	$\sim p$	$\sim \boldsymbol{p} \wedge \boldsymbol{q}$
T	T	F	**F**
T	F	F	**F**
F	T	T	**T**
F	F	T	**F**

1. **Step 1:** Because the statement $\sim (p \wedge \sim p)$ only contains one basic proposition, the truth table will only contain two rows. Statement p may be either true or false.

Step 2: The columns will include p, $\sim p$, $p \wedge \sim p$, and $\sim (p \wedge \sim p)$. Evaluate column $p \wedge \sim p$ with the *and* connective, because the symbol $\wedge$ represents a conjunction or logical *and* statement. True and false is false, and false and true is also false.
Step 3: The final column is the negation of each entry in the third column, both of which are false, so the negation of false is true. Because all the truth values in the final column are true, the statement $\sim (p \wedge \sim p)$ is valid.

p	$\sim p$	$p \wedge \sim p$	$\sim (p \wedge \sim p)$
T	F	F	**T**
F	T	F	**T**

Key Terms

- Truth table
- Multiplication principle
- Valid

Key Concepts

- Determine the true values of logical statements involving negations, conjunctions, and disjunctions.
  - The negation of a logical statement has the opposite true value of the original statement.
- A conjunction is true when both p and q are true, otherwise it is false.
- A disjunction is false when both p and q are false, otherwise it is true.

- Know how to construct a truth table involving negations, conjunctions, and disjunctions and apply the dominance of connectives to determine the truth value of a compound logical statement containing, negations, conjunctions, and disjunctions.

Negation		Conjunction (AND)		Disjunction (OR)					
p	$\sim p$		p	q	$p \wedge q$		p	q	$p \vee q$
T	F		T	T	T		T	T	T
F	T		T	F	F		T	F	T
			F	T	F		F	T	T
			F	F	F		F	F	F

- A logical statement is valid if it is always true. Know how to construct a truth table for a compound statement and use it to determine the validity of compound statements involving negations, conjunctions, and disjunctions.

Video

- Logic Part 2: Truth Values of Conjunctions: Is an "AND" statement true or false?
- Logic Part 3: Truth Values of Disjunctions: Is an "OR" statement true or false?
- Logic Part 4: Truth Values of Compound Statements with "and", "or", and "not"
- Logic Part 5: What are truth tables? How do you set them up?
- Logic Part 6: More on Truth Tables and Setting Up Rows and Column Headings

2.4 Truth Tables for the Conditional and Biconditional

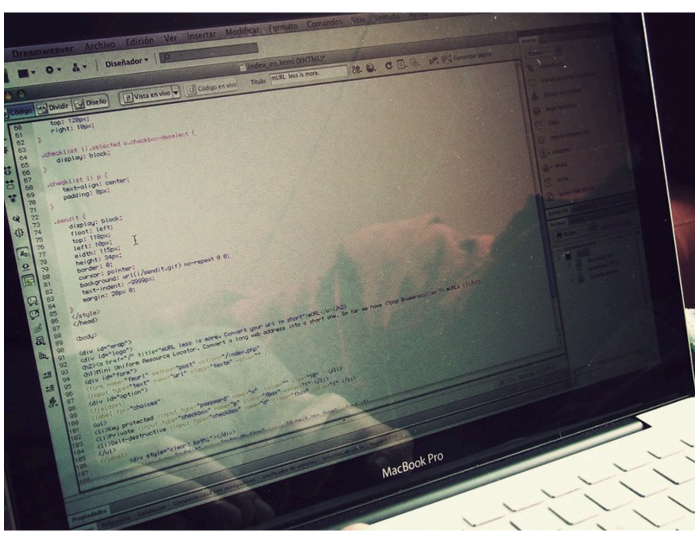


Figure 11: If-then statements use logic to execute directions.

Learning Objectives

After completing this section, you should be able to:

1. Use and apply the conditional to construct a truth table.
2. Use and apply the biconditional to construct a truth table.
3. Use truth tables to determine the validity of conditional and biconditional statements.

   Computer languages use if-then or if-then-else statements as decision statements:

   - *If* the hypothesis is true, *then* do something.

- Or, *if* the hypothesis is true, *then* do something; else do something else.

  For example, the following representation of computer code creates an if-then-else decision statement:

  Check value of variable i.

  If $i < 1$, then print "Hello, World!" else print "Goodbye".

  In this imaginary program, the if-then statement evaluates and acts on the value of the variable i. For instance, if $i = 0$, the program would consider the statement $i < 1$ as true and "Hello, World!" would appear on the computer screen. If instead, $i = 3$, the program would consider the statement $i < 1$ as false (because 3 is greater than 1), and print "Goodbye" on the screen.

  In this section, we will apply similar reasoning without the use of computer programs.

> **PEOPLE IN MATHEMATICS**
>
> The Countess of Lovelace, Ada Lovelace, is credited with writing the first computer program. She wrote an algorithm to work with Charles Babbage's Analytical Engine that could compute the Bernoulli numbers in 1843. In doing so, she became the first person to write a program for a machine that would produce more than just a simple calculation. The computer programming language ADA is named after her.
>
> Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 34 Ada Lovelace: English (1815-1852)" pp. 272-278, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Use and Apply the Conditional to Construct a Truth Table

A **conditional** is a logical statement of the form if p, then q. The conditional statement in logic is a promise or contract. The only time the conditional, $p \rightarrow q$, is false is when the contract or promise is broken.

For example, consider the following scenario. A child's parent says, "If you do your homework, then you can play your video games." The child really wants to play their video games, so they get started right away, finish within an hour, and then show their parent the completed homework. The parent thanks the child for doing a great job on their homework and allows them to play video games. Both the parent and child are happy. The contract was satisfied; true implies true is true.

Now, suppose the child does not start their homework right away, and then struggles to complete it. They eventually finish and show it to their parent. The parent again thanks the child for completing their homework, but then informs the child that it is too late in the evening to play video games, and that they must begin to get ready for bed. Now, the child is really upset. They held up their part of the contract, but they did not receive the promised reward. The contract was broken; true implies false is false.

So, what happens if the child does not do their homework? In this case, the hypothesis is false. No contract has been entered, therefore, no contract can be broken. If the conclusion is false, the child does not get to play video games and might not be happy, but this outcome is expected because the child did not complete their end of the bargain. They did not complete their homework. False implies false is true. The last option is not as intuitive. If the parent lets the child play video games, even if they did not do their homework, neither parent nor child are going to be upset. False implies true is true.

The truth table for the conditional statement below summarizes these results.

p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

CHECKPOINT

Notice that the only time the conditional statement, $p \to q$, is false is when the hypothesis, p, is true and the conclusion, q, is false.

VIDEO

Logic Part 8: The Conditional and Tautologies

Example 1 **Constructing Truth Tables for Conditional Statements**

Assume both of the following statements are true: p: My sibling washed the dishes, and q: My parents paid them $5.00. Create a truth table to determine the truth value of each of the following conditional statements.

1. $p \to q$
2. $p \to \ \sim q$
3. $\sim p \to q$

Solution

1. Because p is true and q is true, the statement $p \to q$ is, “If my sibling washed the dishes, then my parents paid them $5.00.” My sibling did wash the dishes, since p

is true, and the parents did pay the sibling $5.00, so the contract was entered and completed. The conditional statement is true, as indicated by the truth table representing this case:
$T \longrightarrow T = T$.

p	q	$p \rightarrow q$
T	T	**T**

2. $p \rightarrow \sim q$ translates to the statement, “If my sibling washed the dishes, then my parents did not pay them $5.00.” p is true, but $\sim q$ is false. The sibling completed their end of the contract, but they did not get paid. The contract was broken by the parents. The conditional statement is false, as indicated by the truth table representing this case:
$T \longrightarrow F = F$.

p	q	$\sim q$	$p \rightarrow \ \sim q$
T	T	F	**F**

3. $\sim p \rightarrow q$ translates to the statement, “If my sibling did not wash the dishes, then my parents paid them $5.00.” $\sim p$ is false, but q is true. The sibling did not do the dishes. No contract was entered, so it could not be broken. The parents decided to pay them $5.00 anyway. The conditional statement is true, as indicated by the truth table representing this case: $F \longrightarrow T = T$.

p	q	$\sim p$	$\sim p \rightarrow q$
T	T	F	**T**

Example 2 **Determining Validity of Conditional Statements**

Construct a truth table to analyze all possible outcomes for each of the following statements then determine whether they are valid.

1. $p \wedge q \rightarrow \sim q$
2. $p \rightarrow \sim p \vee q$

Solution

1. Applying the dominance of connectives, the statement $p \wedge q \rightarrow \ \sim q$ is equivalent to $(p \wedge q) \rightarrow (\sim q)$. So, the columns of the truth table will include p, q, $p \wedge q$, $\sim q$, and $p \wedge q \rightarrow \ \sim q$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge q \to \sim q$
T	T	T	F	**F**
T	F	F	T	**T**
F	T	F	F	**T**
F	F	F	T	**T**

2. Applying the dominance of connectives, the statement $p \to \ \sim p \vee q$ is equivalent to $(p) \to \ ((\sim p) \vee q)$. So, the columns of the truth table will include p, q, $\sim p$, $\sim p \vee q$, and $p \to \ (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$\sim p$	$\sim p \vee q$	$p \to (\sim p \vee q)$
T	T	F	T	**T**
T	F	F	F	**F**
F	T	T	T	**T**
F	F	T	T	**T**

Use and Apply the Biconditional to Construct a Truth Table

The biconditional, $p \leftrightarrow q$, is a two way contract; it is equivalent to the statement $(p \to q) \wedge (q \to p)$. A biconditional statement, $p \leftrightarrow q$, is true whenever the truth value of the hypothesis matches the truth value of the conclusion, otherwise it is false.

The truth table for the biconditional is summarized below.

p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

VIDEO

Logic Part 11B: Biconditional and Summary of Truth Value Rules in Logic

Example 3 **Constructing Truth Tables for Biconditional Statements**

Assume both of the following statements are true: p: The plumber fixed the leak, and q: The homeowner paid the plumber $150.00. Create a truth table to determine the truth value of each of the following biconditional statements.

1. $p \leftrightarrow q$
2. $p \leftrightarrow \ \sim q$
3. $\sim p \leftrightarrow \ \sim q$

Solution

1. Because p is true and q is true, the statement $p \leftrightarrow q$ is "The plumber fixed the leak if and only if the homeowner paid them $150.00." Because both p and q are true, the leak was fixed and the plumber was paid, meaning both parties satisfied their end of the bargain. The biconditional statement is true, as indicated by the truth table representing this case: $\text{T} \leftrightarrow \text{T} = \text{T}$.

p	q	$p \leftrightarrow q$
T	T	**T**

2. $p \rightarrow \sim q$ translates to the statement, "The plumber fixed the leak if and only if the homeowner did not pay them $150." If the plumber fixed the leak and the homeowner did not pay them, the homeowner will have broken their end of the contract. The biconditional statement is false, as indicated by the truth table representing this case: $\text{T} \leftrightarrow \text{F} = \text{F}$.

p	q	$\sim q$	$p \leftrightarrow \ \sim q$
T	T	F	**F**

3. $\sim p \leftrightarrow \ \sim q$ translates to the statement, "The plumber did not fix the leak if and only if the homeowner did not pay them $150." In this case, neither party—the plumber nor the homeowner—entered into the contract. The leak was not repaired, and the plumber was not paid. No agreement was broken. The biconditional statement is true, as indicated by the truth table representing this case: $\text{F} \leftrightarrow \text{F} = \text{T}$.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \ \sim q$
T	T	F	F	**T**

CHECKPOINT

The biconditional, $p \leftrightarrow q$, is true whenever the truth values of p and q match, otherwise it is false.

VIDEO

Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

Example 4 **Determining Validity of Biconditional Statements**

Construct a truth table to analyze all possible outcomes for each of the following statements, then determine whether they are valid.

1. $p \wedge q \leftrightarrow p \wedge \sim q$
2. $p \vee q \leftrightarrow \sim p \vee q$
3. $p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
4. $p \wedge q \rightarrow \sim r \leftrightarrow p \wedge q \wedge r$

Solution

1. Applying the dominance of connectives, the statement $p \wedge q \leftrightarrow \ \ p \wedge \sim q$ is equivalent to $(p \wedge q) \leftrightarrow \ \ (p \wedge (\sim q))$. So, the columns of the truth table will include p, q, $p \wedge q$, $\sim q$, $p \wedge \sim q$ and $(p \wedge q) \leftrightarrow (p \wedge \sim q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \wedge q$	$\sim q$	$p \wedge \sim q$	$(p \wedge q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	**F**
T	F	F	T	T	**F**
F	T	F	F	F	**T**
F	F	F	T	F	**T**

2. Applying the dominance of connectives, the statement $p \vee q \leftrightarrow \sim p \vee q$ is equivalent to $(p \vee q) \leftrightarrow \ \ ((\sim p) \vee q)$. So, the columns of the truth table will include p, q, $p \vee q$, $\sim p$, $\sim p \vee q$, and $(p \vee q) \leftrightarrow (\sim p \vee q)$. Because there are only two basic propositions, p and q, the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	$p \vee q$	$\sim p$	$\sim p \vee q$	$(p \vee q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	T	F	F	**F**
F	T	T	T	T	**T**
F	F	F	T	T	**F**

3. Applying the dominance of connectives, the statement $p \rightarrow q \leftrightarrow \ \ \sim q \rightarrow \sim p$ is equivalent to $(p \rightarrow q) \leftrightarrow \ \ ((\sim q) \rightarrow (\sim p))$. So, the columns of the truth table will include

p, q, $p \to q$, $\sim q$, $\sim p$, $\sim q \to \sim p$, and $(p \to q) \leftrightarrow (\sim q \to \ \sim p)$. Because there are only two basic propositions, p and q the table will have $2(2) = 4$ rows of truth values to account for all the possible outcomes. The statement is valid because the last column is all true.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to \boldsymbol{q}) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	T	**T**
F	F	T	T	T	T	**T**

4. Applying the dominance of connectives, the statement $p \wedge q \to \ \sim r \ \leftrightarrow \ p \wedge q \wedge r$ is equivalent to $((p \wedge q) \to (\sim r)) \ \leftrightarrow \ ((p \wedge q) \wedge r)$. So, the columns of the truth table will include p, q, r, $\sim r$, $p \wedge q$, $(p \wedge q) \wedge \ \sim r$, $(p \wedge q) \wedge r$, and $((p \wedge q) \to (\sim r)) \ \leftrightarrow \ ((p \wedge q) \wedge r)$. Because there are three basic propositions, p, q, and r, the table will have $2(2)(2) = 8$ rows of truth values to account for all the possible outcomes. The statement is not valid because the last column is not all true.

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \to \sim r$	$(p \wedge q) \wedge r$	$(p \wedge q \to \sim r) \leftrightarrow (p \wedge q \wedge r)$
T	T	T	F	T	F	T	**T**
T	T	F	T	T	T	F	**T**
T	F	T	F	F	T	F	**F**
T	F	F	T	F	T	F	**F**
F	T	T	F	F	T	F	**F**
F	T	F	T	F	T	F	**F**
F	F	T	F	F	T	F	**F**
F	F	F	T	F	T	F	**F**

Key Concepts

- The conditional statement, if p then q, is like a contract. The only time it is false is when the contract has been broken. That is, when p is true, and q is false.

Conditional		
p	q	$p \to q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

- The biconditional statement, p if and only if q, it true whenever p and q have matching true values, otherwise it is false.

Biconditional		
p	q	$p \leftrightarrow q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

- Know how to construct truth tables involving conditional and biconditional statements.
- Use truth tables to analyze conditional and biconditional statements and determine their validity.

Video

- Logic Part 8: The Conditional and Tautologies
- Logic Part 11B Biconditional and Summary of Truth Value Rules in Logic
- Logic Part 13: Truth Tables to Determine if Argument is Valid or Invalid

2.5 Equivalent Statements

Figure 12: How your logical argument is stated affects the response, just like how you speak when holding a conversation can affect how your words are received.

Learning Objectives

After completing this section, you should be able to:

1. Determine whether two statements are logically equivalent using a truth table.
2. Compose the converse, inverse, and contrapositive of a conditional statement

Have you ever had a conversation with or sent a note to someone, only to have them misunderstand what you intended to convey? The way you choose to express your ideas can be as, or even more, important than what you are saying. If your goal is to convince someone that what you are saying is correct, you will not want to alienate them by choosing your words poorly.

Logical arguments can be stated in many different ways that still ultimately result in the same valid conclusion. Part of the art of constructing a persuasive argument is knowing how to arrange the facts and conclusion to elicit the desired response from the intended audience.

In this section, you will learn how to determine whether two statements are **logically equivalent** using truth tables, and then you will apply this knowledge to compose logically equivalent forms of the conditional statement. Developing this skill will provide the additional skills and knowledge needed to construct well-reasoned, persuasive arguments that can be customized to address specific audiences.

> **CHECKPOINT**
> *An alternate way to think about logical equivalence is that the truth values have to match. That is, whenever p is true, q is also true, and whenever p is false, q is also false.*

Determine Logical Equivalence

Two statements, p and q, are logically equivalent when $p \leftrightarrow q$ is a valid argument, or when the last column of the truth table consists of only true values. When a logical statement is always true, it is known as a **tautology**. To determine whether two statements p and q are logically equivalent, construct a truth table for $p \leftrightarrow q$ and determine whether it valid. If the last column is all true, the argument is a tautology, it is valid, and p is logically equivalent to q otherwise, p is not logically equivalent to q.

Example 1 **Determining Logical Equivalence with a Truth Table**

Create a truth table to determine whether the following compound statements are logically equivalent.

1. $p \to q; \sim p \to \ \sim q$
2. $p \to q; \sim p \vee q$

Solution

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, $(p \to q) \leftrightarrow (\sim p \to \ \sim q)$.

p	q	$p \to q$	$\sim p$	$\sim q$	$\sim p \to \ \sim q$	$(p \to q) \leftrightarrow (\sim p \to \ \sim q)$
T	T	T	F	F	T	**T**
T	F	F	F	T	T	**F**
F	T	T	T	F	F	**F**
F	F	T	T	T	T	**T**

Because the last column it not all true, the biconditional is not valid and the statement $p \to q$ is not logically equivalent to the statement $\sim p \to \ \sim q$.

1. Construct a truth table for the biconditional formed by using the first statement as the hypothesis and the second statement as the conclusion, $(p \to q) \leftrightarrow (\sim p \vee q)$.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	**T**
T	F	F	F	F	**T**
F	T	T	T	T	**T**
F	F	T	T	T	**T**

Because the last column is true for every entry, the biconditional is valid and the statement $p \to q$ is logically equivalent to the statement $\sim p \vee q$. Symbolically, $p \to q \ \equiv \ \sim p \vee q$.

Compose the Converse, Inverse, and Contrapositive of a Conditional Statement

The **converse**, **inverse**, and **contrapositive** are variations of the conditional statement, $p \rightarrow q$.

- The converse is if q then p, and it is formed by interchanging the hypothesis and the conclusion. The converse is logically equivalent to the inverse.
- The inverse is if $\sim p$ then $\sim q$, and it is formed by negating both the hypothesis and the conclusion. The inverse is logically equivalent to the converse.
- The contrapositive is if $\sim q$ then $\sim p$, and it is formed by interchanging and negating both the hypothesis and the conclusion. The contrapositive is logically equivalent to the conditional.

The table below shows how these variations are presented symbolically.

	Conditional	Contrapositive	Converse	Inverse			
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$q \rightarrow p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Example 2 **Writing the Converse, Inverse, and Contrapositive of a Conditional Statement**

Use the statements, p: Harry is a wizard and q: Hermione is a witch, to write the following statements:

1. Write the conditional statement, $p \rightarrow q$, in words.
2. Write the converse statement, $q \rightarrow p$, in words.
3. Write the inverse statement, $\sim p \rightarrow \sim q$, in words.
4. Write the contrapositive statement, $\sim q \rightarrow \sim p$, in words.

Solution

1. The conditional statement takes the form, "if p, then q," so the conditional statement is: "If Harry is a wizard, then Hermione is a witch." Remember the *if … then* … words are the connectives that form the conditional statement.
2. The converse swaps or interchanges the hypothesis, p, with the conclusion, q. It has the form, "if q, then p." So, the converse is: "If Hermione is a witch, then Harry is a wizard."
3. To construct the inverse of a statement, negate both the hypothesis and the conclusion. The inverse has the form, "if $\sim p$, then $\sim q$," so the inverse is: "If Harry is not a wizard, then Hermione is not a witch."

4. The contrapositive is formed by negating and interchanging both the hypothesis and conclusion. It has the form, "if $\sim q$, then $\sim p$," so the contrapositive statement is: "If Hermione is not a witch, then Harry is not a wizard."

Example 3 **Identifying the Converse, Inverse, and Contrapositive**

Use the conditional statement, "If all dogs bark, then Lassie likes to bark," to identify the following.

1. Write the hypothesis of the conditional statement and label it with a p.
2. Write the conclusion of the conditional statement and label it with a q.
3. Identify the following statement as the converse, inverse, or contrapositive: "If Lassie likes to bark, then all dogs bark."
4. Identify the following statement as the converse, inverse, or contrapositive: "If Lassie does not like to bark, then some dogs do not bark."
5. Which statement is logically equivalent to the conditional statement?

Solution

1. The hypothesis is the phrase following the *if*. The answer is p: All dogs bark. Notice, the word *if* is not included as part of the hypothesis.
2. The conclusion of a conditional statement is the phrase following the *then*. The word *then* is not included when stating the conclusion. The answer is: q: Lassie likes to bark.
3. "Lassie likes to bark" is q and "All dogs bark" is p. So, "If Lassie likes to bark, then all dogs bark," has the form "if q, then p," which is the form of the converse.
4. "Lassie does not like to bark" is $\sim q$ and "Some dogs do not bark" is $\sim p$. The statement, "If Lassie does not like to bark, then some dogs do not bark," has the form "if $\sim q$, then $\sim p$," which is the form of the contrapositive.
5. The contrapositive $\sim q \rightarrow \ \sim p$ is logically equivalent to the conditional statement $p \rightarrow q$.

Example 4 **Determining the Truth Value of the Converse, Inverse, and Contrapositive**

Assume the conditional statement, $p \rightarrow q$: "If Chadwick Boseman was an actor, then Chadwick Boseman did not star in the movie *Black Panther*" is false, and use it to answer the following questions.

1. Write the converse of the statement in words and determine its truth value.
2. Write the inverse of the statement in words and determine its truth value.

3. Write the contrapositive of the statement in words and determine its truth value.

Solution

1. The only way the conditional statement can be false is if the hypothesis, p: Chadwick Boseman was an actor, is true and the conclusion, q: Chadwick Boseman did not star in the movie *Black Panther*, is false. The converse is $q \rightarrow p$, and it is written in words as: "If Chadwick Boseman did not star in the movie *Black Panther*, then Chadwick Boseman was an actor." This statement is true, because false → true is true.
2. The inverse has the form "$\sim p \rightarrow \ \sim q$." The written form is: "If Chadwick Boseman was not an actor, then Chadwick Boseman starred in the movie *Black Panther*." Because p is true, and q is false, $\sim p$ is false, and $\sim q$ is true. This means the inverse is false → true, which is true. Alternatively, from Question 1, the converse is true, and because the inverse is logically equivalent to the converse it must also be true.
3. The contrapositive is logically equivalent to the conditional. Because the conditional is false, the contrapositive is also false. This can be confirmed by looking at the truth values of the contrapositive statement. The contrapositive has the form "$\sim q \rightarrow \ \sim p$." Because q is false and p is true, $\sim q$ is true and $\sim p$ is false. Therefore, $\sim q \rightarrow \ \sim p$ is true → false, which is false. The written form of the contrapositive is "If Chadwick Boseman starred in the movie *Black Panther*, then Chadwick Boseman was not an actor."

Key Terms

- logically equivalent
- tautology
- inverse
- converse
- contrapositive

Key Concepts

- Two statements p and q are logically equivalent if the biconditional statement, $p \leftrightarrow q$ is a valid argument. That is, the last column of the truth table consists of only true values. In other words, $p \leftrightarrow q$ is a tautology. Symbolically, p is logically equivalent to q is written as: $p \equiv q$.
- A logical statement is a tautology if it is always true.
- To be valid a local argument must be a tautology. It must always be true.
- Know the variations of the conditional statement, be able to determine their truth values and compose statements with them.
  - The converse of a conditional statement, if p then q, is the statement formed by interchanging the hypothesis and conclusion. It is the statement if q then p.

- The inverse of a conditional statement if formed by negating the hypothesis and the conclusion of the conditional statement.
- The contrapositive negates and interchanges the hypothesis and the conclusion.

	Conditional	Contrapositive	Converse	Inverse			
p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

- The conditional statement is logically equivalent to the contrapositive.
- The converse is logically equivalent to the inverse.
- Know how to construct and use truth tables to determine whether statements are logically equivalent.

2.6 De Morgan's Laws

Figure 13: De Morgan's Laws were key to the rise of logical mathematical expression and helped serve as a bridge for the invention of the computer.

Learning Objectives

After completing this section, you should be able to:

1. Use De Morgan's Laws to negate conjunctions and disjunctions.
2. Construct the negation of a conditional statement.
3. Use truth tables to evaluate De Morgan's Laws.

The contributions to logic made by Augustus De Morgan and George Boole during the 19th century acted as a bridge to the development of computers, which may be the greatest invention of the 20th century. **Boolean** logic is the basis for computer science and digital electronics, and without it the technological revolution of the late 20th and early 21st centuries

—including the creation of computer chips, microprocessors, and the Internet—would not have been possible. Every modern computer language uses Boolean logic statements, which are translated into commands understood by the underlying electronic circuits enabling computers to operate. But how did this logic get its name?

PEOPLE IN MATHEMATICS

George Boole

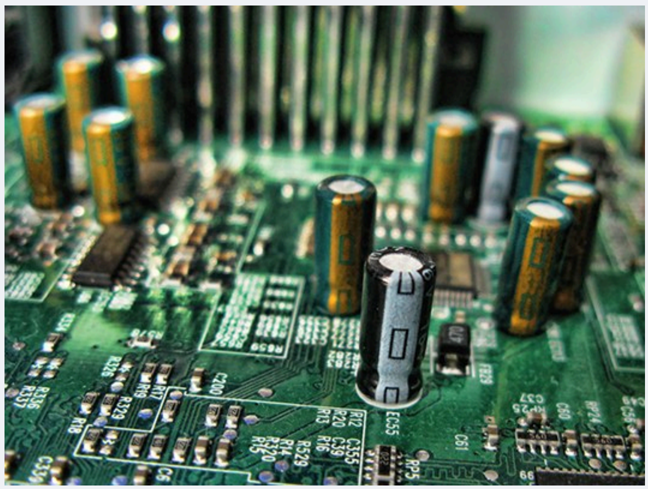


Figure 14: Boole's algebra of logic was foundational in the design of digital computer circuits.

George Boole was born in Lincolnshire, England in 1815. He was the son of a cobbler who provided him some initial education, but Boole was mostly self-taught. He began teaching at 16 years of age, and opened his own school at the age of 20. In 1849, at the age of 34, he was appointed Professor of Mathematics at Queens College in Cork, Ireland. In 1853, he published the paper, *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, which is the treatise that the field of Boolean algebra and digital circuitry was built on.

Reference: Posamentier, Alfred and Spreitzer Christian, "Chapter 35 George Boole: English (1815-1864)" pp. 279-283, *Math Makers: The Lives and Works of 50 Famous Mathematicians*, Prometheus Books, 2019.

Negation of Conjunctions and Disjunctions

In Chapter 1, used a Venn diagram to prove **De Morgan's Law** for set complement over union. Because the complement of a set is analogous to negation and union is analogous to an *or* statement, there are equivalent versions of De Morgan's Laws for logic.

FORMULA

De Morgan's Law for negation of a conjunction: $\sim (p \wedge q) \equiv \ \sim p \vee \sim q$

De Morgan's Law for the negation of a disjunction: $\sim (p \vee q) \equiv \ \sim p \wedge \sim q$

Negation of a conditional: $\sim (p \to q) \equiv p \wedge \sim q$
Writing conditional as a disjunction: $p \to q \equiv \sim p \vee q$

CHECKPOINT
Recall that the symbol for logical equivalence is: $\equiv$.

De Morgan's Laws allow us to write the negation of conjunctions and disjunctions without using the phrase, "It is not the case that …" to indicate the parentheses. Avoiding this phrase often results in a written or verbal statement that is clearer or easier to understand.

Example 1 **Applying De Morgan's Law for Negation of Conjunctions and Disjunctions**

Write the negation of each statement in words without using the phrase, "It is not the case that."

1. Kristin is a biomedical engineer and Thomas is a chemical engineer.
2. A person had cake or they had ice cream.

Solution

1. Kristin is a biomedical engineer and Thomas is a chemical engineer has the form "$p \wedge q$," where p is the statement, "Kristin is a biomedical engineer," and q is the statement, "Thomas is a chemical engineer." By De Morgan's Law, the negation of a conjunction, $\sim (p \wedge q)$, is logically equivalent to $\sim p \vee \sim q$. $\sim p$ is "Kristen is not a biomedical engineer," and $\sim q$ is "Thomas is not a chemical engineer." By De Morgan's Law, the solution has the form "$\sim p \vee \sim q$," so the answer is: "Kristin is not a biomedical engineer or Tom is not a chemical engineer."
2. A person had cake or they had ice cream has the form "$p \vee q$," where p is the statement, "A person had cake," and q is the statement, "A person had ice cream." By De Morgan's Law for the negations of a disjunction, $\sim (p \vee q) \equiv \sim p \wedge \sim q$. The solution is the statement: "A person did not have cake and they did not have ice cream."

Negation of a Conditional Statement

The negation of any statement has the opposite truth values of the original statement. The **negation of a conditional**, $\sim (p \to q)$, is the conjunction of p and not q, $p \wedge \sim q$. Consider the truth table below for the negation of the conditional.

p	q	$p \to q$	$\sim(p \to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

The only time the negation of the conditional statement is true is when p is true, and q is false. This means that $\sim(p \to q)$ is logically equivalent to $p \wedge \sim q$, as the following truth table demonstrates.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$	$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	F	**T**
T	F	F	T	T	T	**T**
F	T	T	F	F	F	**T**
F	F	T	F	T	F	**T**

Example 2 **Constructing the Negation of a Conditional Statement**

Write the negation of each conditional statement.

1. If Adele won a Grammy, then she is a singer.
2. If Henrik Lundqvist played professional hockey, then he did not win the Stanley Cup.

Solution

1. The negation of the conditional statement, $p \to q$, is the statement, $p \wedge \sim q$. The hypothesis of the conditional statement is p: "Adele won a Grammy," and conclusion of the conditional statement is q: "Adele is a singer." The negation of the conclusion, $\sim q$, is the statement: "She is not a singer." Therefore, the answer is $p \wedge \sim q$: "Adele won a Grammy, and she is not a singer."
2. The hypothesis is p: "Henrik Lundqvist played professional hockey," and the conclusion of the conditional statement is q: "He did not win the Stanley Cup." The negation of q is the statement: "He won the Stanley Cup." The negation of the conditional statement is equal to $p \wedge \sim q$: "Henrick Lundqvist played professional hockey, and he won the Stanley Cup."

Example 3 **Constructing the Negation of a Conditional Statement with Quantifiers**

Write the negation of each conditional statement.

1. If all cats purr, then my partner's cat purrs.

2. If a penguin is a bird, then some birds do not fly.

Solution

1. The negation of the conditional statement $p \rightarrow q$ is the statement $p \wedge \sim q$. The hypothesis of the conditional statement is p: "All cats purr," and the conclusion of the conditional statement is q: "My partner's cat purrs." The negation of the conclusion, $\sim q$, is the statement: "My partner's cat does not purr." Therefore, the answer is $p \wedge \sim q$: "All cats purr, but my partner's cat does not purr."
2. The hypothesis is p: "A penguin is a bird," and the conclusion of the conditional statement is q: "Some birds do not fly." The negation of q is the statement: "All birds fly." Therefore, the negation of the conditional statement is equal to $p \wedge \sim q$: "A penguin is a bird, and all birds fly."

Many of the properties that hold true for number systems and sets also hold true for logical statements. The following table summarizes some of the most useful properties for analyzing and constructing logical arguments. These properties can be verified using a truth table.

Property	Conjunction (AND)	Disjunction (OR)
Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \wedge r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Conditional	$\sim (p \rightarrow q) \equiv p \wedge \sim q$	$p \rightarrow q \equiv \sim p \vee q$

Example 4 **Negating a Conditional Statement with a Conjunction or Disjunction**

Write the negation of each conditional statement applying De Morgan's Law.

1. If mom needs to buy chips, then Mike had friends over and Bob was hungry.
2. If Juan had pizza or Chris had wings, then dad watched the game.

Solution

1. The conditional has the form "If p then q or r," where p is "Mom needs to buy chips," q is "Mike had friends over," and r is "Bob was hungry." The negation of $p \rightarrow (q \wedge r)$ is $p \wedge \sim (q \wedge r)$. Applying De Morgan's Law to the statement $\sim (q \wedge r)$ the result is $\sim q \vee \sim r$, so our conditional statement becomes $p \wedge (\sim q \vee \sim r)$. By the distributive property for conjunction over disjunction, this statement is equivalent to $(p \wedge \sim q) \vee (p \wedge \sim r)$. Translating the statement $(p \wedge \sim q) \vee (p \wedge \sim r)$ into words, the solution is: "Mom needs to buy chips and Mike did not have friends over, or Mom needs to buy chips and Bob was not hungry."

2. The conditional has the form "If p or q, then r," where p is "Juan had pizza," q is "Chris had wings," and r is "Dad watched the game." The negation of $(p \vee q) \to r$ is $(p \vee q) \wedge \sim r$. By the distributive property for disjunction over conjuction, the statement is equivalent to $(p \ \vee \sim r) \wedge (q \ \vee \sim r)$. Translating the statement $(p \ \vee \sim r) \wedge (q \ \vee \sim r)$ into words, the solution is: "Juan had pizza or dad did not watch the game, and Chris had wings or dad did not watch the game."

Evaluating De Morgan's Laws with Truth Tables

In Chapter 1, you learned that you could prove the validity of De Morgan's Laws using Venn diagrams. Truth tables can also be used to prove that two statements are logically equivalent. If two statements are logically equivalent, you can use the form of the statement that is clearer or more persuasive when constructing a logical argument.

The next example will prove the validity of one of De Morgan's Laws using a truth table. The same procedure can be applied to any two logical statement that you believe are equivalent. If the last column of the truth table is a tautology, then the two statements are logically equivalent.

Example 5 **Verifying De Morgan's Law for Negation of a Conjunction**

Construct a truth table to verify De Morgan's Law for the negation of a conjunction, $\sim (p \wedge q) \ \equiv \ \sim p \vee \sim q$, is valid.

Solution

Step 1: To verify any logical equivalence, you must first replace the logical equivalence symbol, $\equiv$, with the biconditional symbol, $\leftrightarrow$. The statement $\sim (p \wedge q) \ \equiv \ \sim p \ \vee \sim q$ becomes $\sim (p \wedge q) \ \leftrightarrow \ \sim p \ \vee \sim q$.

Step 2: Next, you create a truth table for the statement. Because we have two basic statements, p, and q, the truth table will have four rows to account for all the possible outcomes. The columns will be p, q, $\sim p$, $\sim q$, $p \wedge q$,$\sim (p \wedge q)$,$\sim p \ \vee \sim q$, and the biconditional statement is $\sim (p \wedge q) \leftrightarrow \ \sim p \ \vee \sim q$.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	**T**
T	F	F	T	F	T	T	**T**
F	T	F	T	T	F	T	**T**
F	F	F	T	T	T	T	**T**

Step 3: Finally, verify that the statement is valid by confirming it is a tautology. In this instance, the last column is all true. Therefore, the statement is valid and De Morgan's Law for the negation of a conjunction is verified.

Key Terms

- Boolean logic

- negation of a conditional

Key Concepts

- De Morgan's Law for the negation of a disjunction states that, $\sim (p \vee q)$ is logically equivalent to $\sim p \wedge \sim q$.
- De Morgan's Law The negation of a conjunction states that, $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
- Use De Morgan's Laws to negate conjunctions and disjunctions.
- The negation of a conditional statement, if p then q is logically equivalent to the statement p and not q. Use this property to write the negation of conditional statements.
- Use truth tables to evaluate De Morgan's Laws.

2.7 Logical Arguments

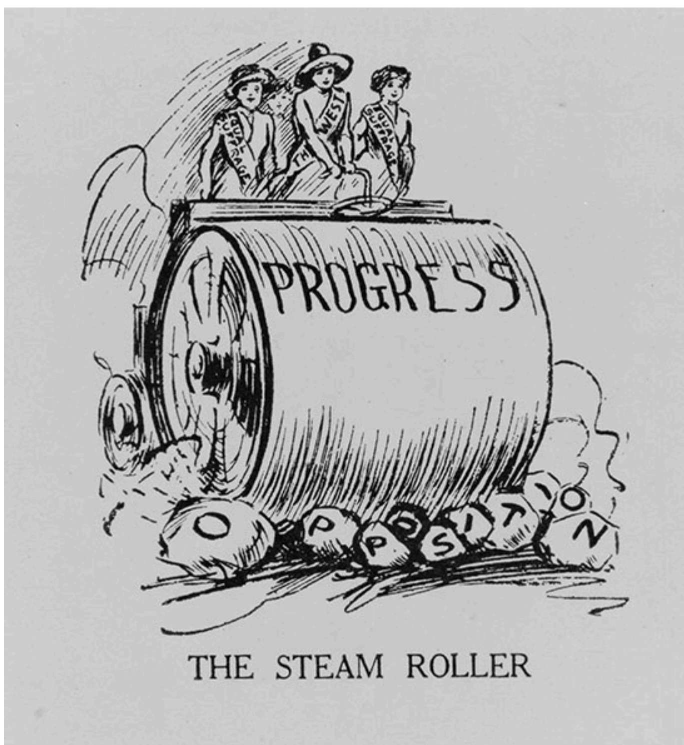


Figure 15: Not all logical arguments are valid, and the ongoing fight for equal rights proves that much progress has yet to be made.

Learning Objectives

After completing this section, you should be able to:

1. Apply the law of detachment to determine the conclusion of a pair of statements.
2. Apply the law of denying the consequent to determine the conclusion for pairs of statements.

3. Apply the chain rule to determine valid conclusions for pairs of true statements.

   The previous sections of this chapter provide the foundational skills for constructing and analyzing logical arguments. All logical arguments include a set of premises that support a claim or conclusion; but not all logical arguments are valid and sound. A logical argument is valid if its conclusion follows from the premises, and it is **sound** if it is valid and all of its premises are true. A false or deceptive argument is called a **fallacy**. Many types of fallacies are so common that they have been named.

> **WHO KNEW?**
>
> In 1936, Dale Carnegie published his first book, titled *How to Win Friends and Influence People*. It was marketed as training materials for the improvement of public speaking and negotiation skills, and the methods it presented are still used today. Carnegie famously said, "When dealing with people, remember you are not dealing with creatures of logic, but creatures of emotion."
>
> People who put forth fallacious logical arguments often take advantage of our susceptibility to emotional appeals, to try to convince us that what they are saying is true. The study of logic helps us combat this weakness through recognition and learning to focus on the facts and structure of the argument.

This section focuses on the two main forms that logical arguments can take. While inductive arguments attempt to draw a more general conclusion from a pattern of specific premises, **deductive arguments** attempt to draw specific conclusions from at least one or more general premises. Deductive arguments can be proven to be valid using Venn diagrams or truth tables.

Inductive arguments generally cannot be proven to be true. They are judged as being strong or weak, but, like any opinion, whether you believe an argument is strong or weak often depends on your knowledge of the topic being discussed along with the evidence being provided in the premises. *Hasty generalization* is the name given to any fallacy that presents a weak inductive argument.

> **CHECKPOINT**
>
> *Be careful! Premises may be true or false. If a premise is false, the claims made by the argument should be questioned.*

Law of Detachment

The **law of detachment** is a valid form of a conditional argument that asserts that if both the conditional, $p \to q$, and the hypothesis, p, are true, then the conclusion q must also be true. The law of detachment is also called affirming the hypothesis (or antecedent) and modus ponens. Symbolically, it has the form $((p \to q) \wedge p) \to q$.

Law of Detachment	
Premise:	$p \rightarrow q$
Premise:	p
Conclusion:	$\therefore \quad q$

CHECKPOINT

The $\therefore$ is read as the word, "therefore."

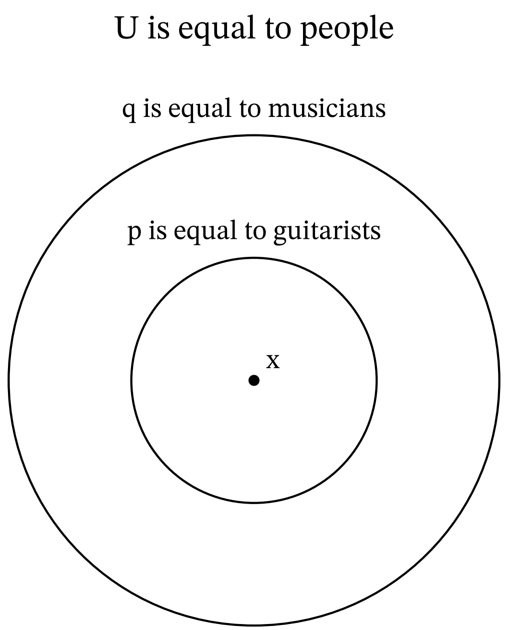
Looking at the truth table for the conditional statement, the only time the conditional is true is when the hypothesis p is also true. The only place this happens is in the first row, where q is also true, confirming that the law of detachment is a valid argument.

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Another way to verify that the law of detachment is a valid argument is to construct a truth table for the argument $((p \rightarrow q) \wedge p) \rightarrow q$ and verify that it is a tautology.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Venn diagrams may also be used to verify deductive arguments, which include conditional premises. Consider the statement $p \rightarrow q$: "If you play guitar, then you are a musician." The set of guitarists is a subset of the set of musicians, $p \subset q$. To verify that an argument is valid using a Venn diagram, draw the Venn diagram representing all the premises in the argument only, as shown. Then verify if the conclusion is also represented by the Venn diagram of the premises. If it is, the argument is valid. If it is not, the argument is not valid. The set of guitarists is drawn as a subset of the set of musicians to represent the premise, $p \rightarrow q$. The $\times$ represents the premise: p is true. This completes the drawing of the premises.



Now, examine the Venn diagram to verify if the conclusion is included in the picture. The conclusion is q. Because the $\times$ is in the set p, and p is a subset of q, $\times$ is also in q therefore, the law of detachment is a valid argument.

CHECKPOINT

Remember that an argument can be valid without being true. For the argument to be proven true, it must be both valid and sound. An argument is sound if all its premises are true.

VIDEO

Logic Part 14: Common Argument Forms like Modus Ponens and Tollens

Example 1 **Applying the Law of Detachment to Determine a Valid Conclusion**

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of detachment to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci was an artist.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan played for the Chicago Bulls.
3. If all fish have gills, then clown fish have gills. All fish have gills.

Solution

1. The premises are $p \to q$: If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*, and p : Leonardo da Vinci was an artist. This argument has the form of the law of detachment, so, the conclusion is q : Leonardo da Vinci painted the *Mona Lisa*.
2. The premises follow the form of the law of detachment, so a valid conclusion would be q. The premises are $p \to q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and p : Michael Jordan played for the Chicago Bulls. The conclusion that follows from the premises is q : Michael Jordan was not a soccer player.
3. The premises are $p \to q$: If all fish have gills, then clown fish have gills, and p : All fish have gills. This argument has the form of the law of detachment, so the conclusion is q : clown fish have gills.

Law of Denying the Consequent

Another form of a valid conditional argument is called the **law of denying the consequent**, or modus tollens. Recall, that the conditional statement, $p \to q$, is logically equivalent to the contrapositive, $\sim q \to \ \sim p$. So, if the conditional statement is true, then the contrapositive statement is also true. By the law of detachment, if $\sim q$ is also true, then it follows that $\sim p$ must also be true. Symbolically, it has the form $((p \to q) \wedge \sim q) \to \ \sim p$.

Law of Denying the Consequent	
Premise:	$p \to q$
Premise:	$\sim q$
Conclusion:	$\therefore \ \sim p$

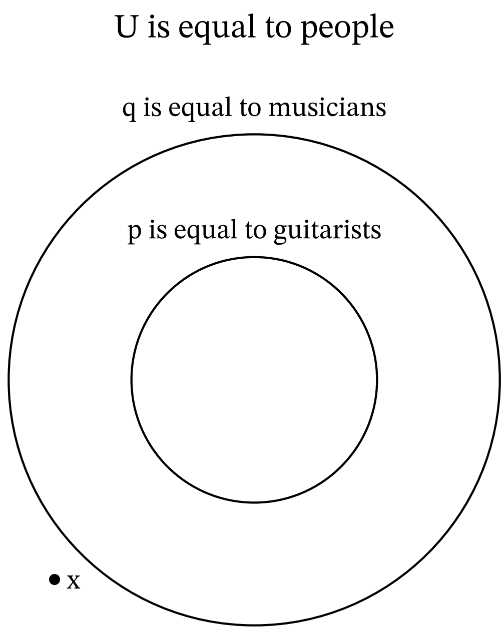
CHECKPOINT

The conditional statement can also be described as, "If antecedent, then consequent."This is where the law of denying the consequent gets its name.

To verify if the law of denying the consequent is a valid argument, construct a truth table for the argument, $((p \to q) \wedge \sim q) \to \ \sim p$, and verify that it is a tautology.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$	$((p \to q) \wedge \sim q) \to \sim p$
T	T	F	F	T	F	**T**
T	F	F	T	F	F	**T**
F	T	T	F	T	F	**T**
F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise: $p \to q$: "If you play guitar, then you are a musician." We will change the second premise to $\sim q$. In this case, the $\times$ represents the premise, $\sim q$. So, it will be placed inside the universal set of all people, but outside the set of musicians, as depicted in the Venn diagram in .



Because the $\times$ is also outside the set of guitarists, the statement $\sim p$ follows from the premises and the argument is valid.

Example 2 **Applying the Law of Denying the Consequent to Determine a Valid Conclusion**

Each pair of statements represents the premises in a logical argument. Based on these premises, apply the law of denying the consequent to determine a valid conclusion.

1. If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*. Leonardo da Vinci did not paint the *Mona Lisa*.
2. If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player. Michael Jordan was a soccer player.
3. If all fish have gills, then clown fish have gills. Clown fish do not have gills.

Solution

1. The premises are $p \to q$: If Leonardo da Vinci was an artist, then he painted the *Mona Lisa*, and $\sim q$: Leonardo da Vinci did not paint the *Mona Lisa*. This argument has the form of the law of denying the consequent, so the conclusion is $\sim p$: Leonardo da Vinci was not an artist.
2. The premises follow the form of the law of denying the consequent, so a valid conclusion would be $\sim p$. The premises are: $p \to q$: If Michael Jordan played for the Chicago Bulls, then Michael Jordan was not a soccer player, and $\sim q$: Michael Jordan

was a soccer player. The conclusion that follows from the premises is $\sim p$: Michael Jordan did not play for the Chicago Bulls.

3. The premises are $p \rightarrow q$: If all fish have gills, then clown fish have gills, and $\sim q$: Clown fish do not have gills. This argument has the form of the law denying the consequent, so the conclusion is $\sim p$: Some fish do not have gills.

Chain Rule for Conditional Arguments

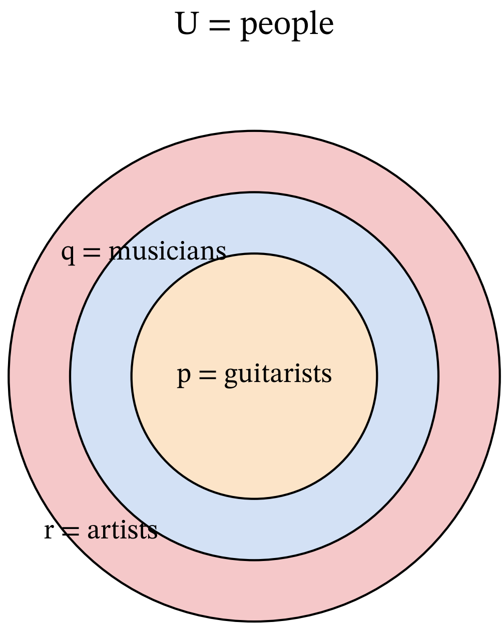
The **chain rule for conditional arguments** is another form of a valid conditional argument. It is also called hypothetical syllogism or the transitivity of implication. Recall that the conditional statement $p \rightarrow q$ can also be read as p implies q. This is where the name *transitivity of implication* comes from. The transitive property for numbers states that, if $3 < 4$ and $4 < 5$, then it follows that $3 < 5$. The chain rule extends this property to conditional statements. If the premises of the argument consist of two conditional statements, with the form "$p \rightarrow q$" and "$q \rightarrow r$," then it follows that $p \rightarrow r$. Symbolically, it has the form $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$.

Chain Rule for Conditional Arguments	
Premise:	$p \rightarrow q$
Premise:	$q \rightarrow r$
Conclusion:	$\therefore \quad p \rightarrow r$

To verify the chain rule for conditional arguments, construct a truth table for the argument, $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$, and verify that it is a tautology.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$	$p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	T	F	T	F	T	**T**
T	F	F	F	T	F	F	**T**
F	T	T	T	T	T	T	**T**
F	T	F	T	F	F	T	**T**
F	F	T	T	T	T	T	**T**
F	F	F	T	T	T	T	**T**

To verify an argument of this form using a Venn diagram, again consider the premise $p \rightarrow q$: "If you play guitar, then you are a musician," but change the second premise to $q \rightarrow r$: "If you are a musician, then you are an artist." In this case, the set p of guitarists is a subset of the set r of artists, and it follows that if you are a guitarist, then you are an artist. Therefore, the conclusion $p \rightarrow r$ follows from the premises and the chain rule for logical arguments is valid.



Example 3 **Applying the Chain Rule for Conditional Arguments to Determine a Valid and Sound Conclusion**

Each pair of statements represents true premises in a logical argument. Based on these premises, apply the chain rule for conditional arguments to determine a valid and sound conclusion.

1. If my roommate goes to work, then my roommate will get paid. If my roommate gets paid, then my roommate will pay their bills.
2. If robins can fly, then some birds can fly. If some birds can fly, then we will watch birds fly.
3. If Irma is a teacher, then Irma has a college degree. If Irma has a college degree, then Irma graduated from college.

Solution

1. The premises are $p \rightarrow q$: "If my roommate goes to work, then they will get paid," and $q \rightarrow r$: "If my roommate gets paid, then my roommate will pay their bills." This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form "$p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If my roommate goes to work, then my roommate will pay their bills."
2. The premises are $p \rightarrow q$: "If robins can fly, then some birds can fly," and $q \rightarrow r$: "If some birds can fly, then we will watch them fly." This argument has the form of the chain rule for conditional arguments, so, the valid conclusion will have the form "$p \rightarrow r$." Because all the premises are true, the valid and sound conclusion of this argument is: "If robins can fly, then we will watch birds fly."

3. The premises are $p \to q$: (see line 1 of solution 1 and 2 above) “If Irma is a teacher, then Irma has a college degree,” and $q \to r$: “If Irma has a college degree, then Irma graduated from college.” This argument has the form of the chain rule for conditional arguments, so the valid conclusion will have the form “$p \to r$.” Because all the premises are true, the valid and sound conclusion of this argument is: “If Irma is a teacher, then Irma graduated from college.”

Projects

Logic Gates

Logic gates are the basis for all digital circuits.

1. Research and document the following terms: logic gate, OR gate, AND gate, and NOT gate.
2. Construct a diagram of a NAND gate, NOR gate, and a XOR gate by using at least two of the following gates: AND, OR, and NOT.
3. Digital electronics use a 1 for true or on, and a 0 for false or off. Create a truth table documenting all possible cases using 0s and 1s for the NAND gate, NOR gate and XOR gate.
4. Use a truth table to explain how XOR is related to the biconditional statement.

Logical Fallacies

Fallacies are false or deceptive logical arguments.

1. Research and document the structure of five of the following named fallacies: hasty generalization, limited choice, false cause, appeal to popularity, appeal to emotion, appeal to authority, personal attack, gamblers’ ruin, slippery slope, and circular reasoning.
2. Create a presentation highlighting one of the five fallacies researched in the previous question. The presentation must include an introductory slide with the title of the fallacy and the form or structure of the argument. The second slide must include an example of this fallacy as used in a commercial, a political cartoon or a current event or new article. The third slide must include an explanation of why the example on slide to is a representative example of the fallacy. The last slide must include citations for any materials used. No textbooks should be used as reference.

Careers in Logic

Lawyers, mathematicians, and computer programmers are a few of the careers that require knowledge of logic.

1. What career are you interested in? Research how knowledge of logic applies to your chosen field of study. Then, write a cover letter for a position in your field you’d like to apply to. In the cover letter, include how your knowledge of logic qualifies you for the position you are applying for. If you do not think logic is important for your given career choice, find a position where logic is an essential element of the position and complete the project by pretending you are writing a cover letter for that job.

Key Terms

- sound
- fallacy
- deductive arguments
- law of detachment
- law of denying the consequent
- chain rule for conditional arguments

Key Concepts

- A logical argument uses a series of facts or premises to justify a conclusion or claim. It is valid if its conclusion follows from the premises, and it is sound if it is valid, and all of its premises are true.
- The law of detachment is a valid form of a conditional argument that asserts that if both the conditional, $p \rightarrow q$is true and the hypothesis, p is true, then the conclusion q must also be true.

Law of Detachment	
Premise:	$p \rightarrow q$
Premise:	p
Conclusion:	$\therefore q$

- Know how to apply the law of detachment to determine the conclusion of a pair of statements.
- Thelaw of denying the consequentis a valid form of a conditional argument that asserts that if both the conditional, $p \rightarrow q$is true and the negation of the conclusion, $\sim q$ is true, then the negation of the hypothesis $\sim p$ must also be true.

Law of Denying the Consequent	
Premise:	$p \rightarrow q$
Premise:	$\sim q$
Conclusion:	$\therefore \sim p$

- Know how to apply the law of denying the consequent to determine the conclusion for pairs of statements.
- The chain rule for conditional arguments is a valid form of a conditional argument that asserts that if the premises of the argument have the form, $p \rightarrow q$and $q \rightarrow r$, then it follows that $p \rightarrow r$.

Chain Rule for Conditional Arguments	
Premise:	$p \to q$
Premise:	$q \to r$
Conclusion:	$\therefore\ p \to r$

- Know how to apply the chain rule to determine valid conclusions for pairs of true statements.

Video

- Logic Part 14: Common Argument Forms like Modus Ponens and Tollens