Chapter 1: Sets

Figure 1: A flatware drawer is like a set in that it contains distinct objects.

Think of a drawer in your kitchen used to store flatware. This drawer likely holds forks, spoons, and knives, and possibly other items such as a meat thermometer and a can opener. The drawer in this case represents a tool used to group a collection of objects. The members of the group are the individual items in the drawer, such as a fork or a spoon.

The members of a set can be anything, such as people, numbers, or letters of the alphabet. In statistical studies, a **set** is a well-defined collection of objects used to identify an entire population of interest. For example, in a research study examining the effects of a new medication, there can be two sets of people: one set that is given the medication and a different set that is given a placebo (control group). In this chapter, we will discuss sets and Venn diagrams, which are graphical ways to show relationships between different groups.

1.1 Basic Set Concepts

Figure 2: A spoon, fork, and knife are elements of the set of flatware.

Learning Objectives

After completing this section, you should be able to:

1. Represent sets in a variety of ways.
2. Represent well-defined sets and the empty set with proper set notation.
3. Compute the cardinal value of a set.
4. Differentiate between finite and infinite sets.
5. Differentiate between equal and equivalent sets.

Sets and Ways to Represent Them

Think back to your kitchen organization. If the drawer is the **set**, then the forks and knives are **elements** in the set. Sets can be described in a number of different ways: by roster, by set-builder notation, by interval notation, by graphing on a number line, and by Venn diagrams. Sets are typically designated with capital letters. The simplest way to represent a set with only a few members is the **roster** (or *listing*) **method**, in which the elements in a set are listed, enclosed by curly braces and separated by commas. For example, if F represents our set of flatware, we can represent F by using the following set notation with the roster method:

$$F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$$

Example 1 **Writing a Set Using the Roster or Listing Method**

Write a set consisting of your three favorite sports and label it with a capital S.

Solution

There are multiple possible answers depending on what your three favorite sports are, but any answer must list three different sports separated by commas, such as the following:

$$S = \{\text{hockey, basketball, soccer}\}$$

All the sets we have considered so far have been **well-defined sets**. A well-defined set clearly communicates whether an element is a member of the set or not. The members of a well-defined set are fixed and do not change over time. Consider the following question. What are your top 10 songs of 2021? You could create a list of your top 10 favorite songs from 2021, but the list your friend creates will not necessarily contain the same 10 songs. So, the set of your top 10 songs of 2021 is not a well-defined set. On the other hand, the set of the letters in your name is a well-defined set because it does not vary (unless of course you change your name). The NFL wide receiver, Chad Johnson, famously changed his name to Chad Ochocinco to match his jersey number of 85.

Example 2 **Identifying Well-Defined Sets**

For each of the following collections, determine if it represents a well-defined set.

1. The group of all past vice presidents of the United States.
2. A group of old cats.

Solution

1. The group of all past vice presidents of the United States is a well-defined set, because you can clearly identify if any individual was or was not a member of that group. For example, Britney Spears is not a member of this set, but Joe Biden is a member of this set.
2. A group of old cats is not a well-defined set because the word old is ambiguous. Some people might consider a seven-year-old cat to be old, while others might think a cat is not old until it is 13 years old. Because people can disagree on what is and what is not a member of this group, the set is not well-defined.

On January 20, 2021, Kamala Harris was sworn in as the first woman vice president of the United States of America. If we were to consider the set of all women vice presidents of the United States of America prior to January 20, 2021, this set would be known as an **empty set** the number of people in this set is 0, since there were no women vice presidents before Harris. The empty set, also called the null set, is written symbolically using a pair of braces, { }, or a zero with a slash through it, ∅.

CHECKPOINT

The set containing the number $0, \{0\}$, *is a set with one element in it. It is not the same as the empty set,* $\{\ \}$, *which does not have any elements in it. Symbolically:* $\{0\} \neq \{\ \}$.

Example 3 **Representing the Empty Set Symbolically**

Represent each of the following sets symbolically.

1. The set of prime numbers less than 2.
2. The set of birds that are also mammals.

Solution

1. A prime number is a natural number greater than 1 that is only divisible by one and itself. Since there are no prime numbers less than 2, this set is empty, and we can represent it symbolically as follows: $\{\ \}$ or $\varnothing$. These two different symbols for the empty set can be used interchangeably.
2. The set of birds and the set of mammals do not intersect, so the set of birds that are also mammals is empty, and we can represent it symbolically as $\varnothing$ or $\{\ \}$.

WHO KNEW?

The Number Zero

We use the number zero to represent the concept of nothing every day. The machine language of computers is binary, consisting only of zeros and ones, and even way before that, the number zero was a powerful invention that allowed our understanding of mathematics and science to develop. The historical record shows the Babylonians first used zeros around 300 B.C., while the Mayans developed and began using zero separately around 350 A.D. What is considered the first formal use of zero in arithmetic operations was developed by the Indian mathematician Brahmagupta around 650 A.D.

Brahmagupta, Mathematician and Astronomer

Another interesting feature of the number zero is that although it is an even number, it is the only number that is neither negative nor positive.

For larger sets that have a natural ordering, sometimes an ellipsis is used to indicate that the pattern continues. It is common practice to list the first three elements of a set to establish a pattern, write the ellipsis, and then provide the last element. Consider the set of all lowercase letters of the English alphabet, A. This set can be written symbolically as $A = \{a, b, c, \ldots, z\}$.

The sets we have been discussing so far are **finite sets**. They all have a limited or fixed number of elements. We also use an ellipsis for **infinite sets**, which have an unlimited number of elements, to indicate that the pattern continues. For example, in set theory, the set of **natural numbers**, which is the set of all positive counting numbers, is represented as $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Notice that for this set, there is no element following the ellipsis. This is because there is no largest natural number; you can always add one more to get to the next natural number. Because the set of natural numbers grows without bound, it is an infinite set.

Example 4 **Writing a Finite Set Using the Roster Method and an Ellipsis**

Write the set of even natural numbers including and between 2 and 100, and label it with a capital E. Include an ellipsis.

Solution

Write the label, E, followed by an equal sign and then a bracket. Write the first three even numbers separated by commas, beginning with the number two to establish a pattern. Next, write the ellipsis followed by a comma and the last number in the list, 100. Finally, write the closing bracket to complete the set.

Write the label, E, followed by an equal sign and then a bracket.

$$E = \{$$

Write the first three even numbers separated by commas, beginning with the number 2 to establish a pattern.

$$E = \{2, 4, 6,$$

Next, write the ellipsis followed by a comma and the last number in the list, 100.

$$E = \{2, 4, 6, \ldots, 100$$

Finally, write the closing bracket to complete the set.

$$E = \{2, 4, 6, \ldots, 100\}$$

Our number system is made up of several different infinite sets of numbers. The set of **integers**, $\mathbb{Z}$, is another infinite set of numbers. It includes all the positive and negative counting numbers and the number zero. There is no largest or smallest integer.

Example 5 **Writing an Infinite Set Using the Roster Method and Ellipses**

Write the set of integers using the roster method, and label it with a $\mathbb{Z}$.

Solution

Step 1: As always, we write the label and then the opening bracket. Because the negative counting numbers are infinite, to represent that the pattern continues without bound to the left, we must use an ellipsis as the first element in our list.

Step 2: We place a comma and follow it with at least three consecutive integers separated by commas to establish a pattern.

Step 3: Add an ellipsis to the end of the list to show that the set of integers continues without bound to the right.

Complete the list with a closing bracket. The set of integers may be represented as follows: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

A shorthand way to write sets is with the use of **set builder notation**, which is a verbal description or formula for the set. For example, the set of all lowercase letters of the English alphabet, A, written in set builder notation is:

$$A = \{x \mid x \text{ is a lowercase letter of the English alphabet.}\}$$

This is read as, "Set A is the set of all elements x such that x is a lowercase letter of the English alphabet."

Example 6 **Writing a Set Using Set Builder Notation**

Using set builder notation, write the set B of all types of balls. Explain what the notation means.

Solution

The verbal description of the set is, "Set B is the set of all elements b such that b is a ball." This set can be written in set builder notation as follows: $B = \{b \mid b \text{ is a ball .}\}$

Example 7 **Writing Sets Using Various Methods**

Consider the set of letters in the word "happy." Determine the best way to represent this set, and then write the set using either the roster method or set builder notation, whichever is more appropriate.

Solution

Because the letters in the word "happy" consist of a small finite set, the best way to represent this set is with the roster method. Choose a label to represent the set, such as H.

$$H = \{\text{h, a, p, y}\}$$

.

Notice that the letter "p" is only represented one time. This occurs because when representing members of a set, each unique element is only listed once no matter how many times it occurs. Duplicate elements are never repeated when representing members of a set.

Computing the Cardinal Value of a Set

Almost all the sets most people work with outside of pure mathematics are finite sets. For these sets, the **cardinal value** or **cardinality** of the set is the number of elements in the set. For finite set A, the cardinality is denoted symbolically as $n(A)$. For example, a set that contains four elements has a cardinality of 4.

How do we measure the cardinality of infinite sets? The 'smallest' infinite set is the set of natural numbers, or counting numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$. This set has a cardinality of $\aleph_0$ (pronounced "aleph-null"). All sets that have the same cardinality as the set of natural numbers are **countably infinite**. This concept, as well as notation using aleph, was introduced by mathematician Georg Cantor who once said, "A set is a Many that allows itself to be thought of as a One."

Example 8 **Computing the Cardinal Value of a Set**

Write the cardinal value of each of the following sets in symbolic form.

1. $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$
2. The empty set.

Solution

1. There are 5 distinct elements in set F: a fork, a spoon, a knife, a meat thermometer, and a can opener. Therefore, the cardinal value of set F is 5 and written symbolically as $n(F) = 5$.
2. Because the empty set does not have any elements in it, the cardinality of the empty set is zero. Symbolically we write this as: $n(\emptyset) = 0$.

Now that we have learned to represent finite and infinite sets using both the roster method and set builder notation, we should also be able to determine if a set is finite or infinite based on its verbal or symbolic description. One way to determine if a set is finite or not is to determine the cardinality of the set. If the cardinality of a set is a natural number, then the set is finite.

Example 9 **Differentiating Between Finite and Infinite Sets**

Classify each of the following sets as infinite or finite.

1. $E = \{2, 4, 6, 8, 10\}$
2. A is the set of lowercase letters of the English Alphabet, $A = \{\text{a, b, c, ..., z}\}$.
3. $\mathbb{Q} = \{\frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0\}$

Solution

1. $n(E) = 5$. Since 5 is a natural number, the set is finite.
2. $n(A) = 26$. Since 26 is a natural number, the set is finite.
3. Set $\mathbb{Q}$ is the set of rational numbers or fractions. Because the set of integers is a subset of the set of rational numbers, and the set of integers is infinite, the set of rational numbers is also infinite. There is no smallest or largest rational number.

Equal versus Equivalent Sets

When speaking or writing we tend to use equal and equivalent interchangeably, but there is an important distinction between their meanings. Consider a new Ford Escape Hybrid and a new Toyota Rav4 Hybrid. Both cars are hybrid electric sport utility vehicles; in that sense, they are **equivalent**. They will both get you from place to place in a relatively fuel-efficient way. In this example we are comparing the single member set {Toyota Rav4 Hybrid} to the single member set {Ford Escape Hybrid}. Since these two sets have the same number of elements, they are also equivalent mathematically, meaning they have the same cardinality. But they are not equal, because the two cars have different looks and features, and probably even handle differently. Each manufacturer will emphasize the features unique to their vehicle to persuade you to buy it; if the SUVs were truly equal, there would be no reason to choose one over the other.

Now consider two Honda CR-Vs that are made with exactly the same parts, on the same assembly line within a few minutes of each other—these SUVs are **equal**. They are identical to each other, containing the same elements without regard to order, and the only differentiator when making a purchasing decision would be varied pricing at different dealerships. The set {Honda CR-V} is equal to the set {Honda CR-V}. Symbolically, we represent equal sets as $A = B$ and equivalent sets as $A \sim B$.

Now, let us consider a Toyota dealership that has 10 RAV4s on the lot, 8 Prii, 7 Highlanders, and 12 Camrys. There is a one-to-one relationship between the set of vehicles on the lot and the set consisting of the number of each type of vehicle on the lot. Therefore, these two sets are equivalent, but not equal. The set {RAV4, Prius, Highlander, Camry} is equivalent to the set {10, 8, 7, 12} because they have the same number of elements.

VIDEO

Equal and Equivalent Sets

CHECKPOINT

If two sets are equal, they are also equivalent, because equal sets also have the same cardinality.

Example 10 **Differentiating Between Equivalent and Equal Sets**

Determine if the following pairs of sets are equal, equivalent, or neither.

1. $E = \{2, 4, 6, 8, 10\}$ and $F =$ {fork, spoon, knife, meat thermometer, can opener}
2. The empty set and the set of prime numbers less than 2.
3. The set of vowels in the word happiness and the set of consonants in the word happiness.

Solution

1. Sets E and F both have a cardinal value of 5, but the elements in these sets are different. So, the two sets are equivalent, but they are not equal: $E \sim F$.
2. The set of prime numbers consists of the set of counting numbers greater than one that can only be divided evenly by one and itself. The set of prime numbers less than 2 is an empty set, since there are no prime numbers less than 2. Therefore, these two sets are equal (and equivalent).
3. The set of vowels in the word happiness is {a, i, e} and the set of consonants in the word happiness is {h, p, n, s}. The cardinal value of these sets two sets is $n(\{a, i, e\}) = 3$ and $n(\{h, p, n, s\}) = 4$, respectively. Because the cardinality of the two sets differs, they are not equivalent. Further, their elements are not identical, so they are also not equal.

PEOPLE IN MATHEMATICS

Georg Cantor

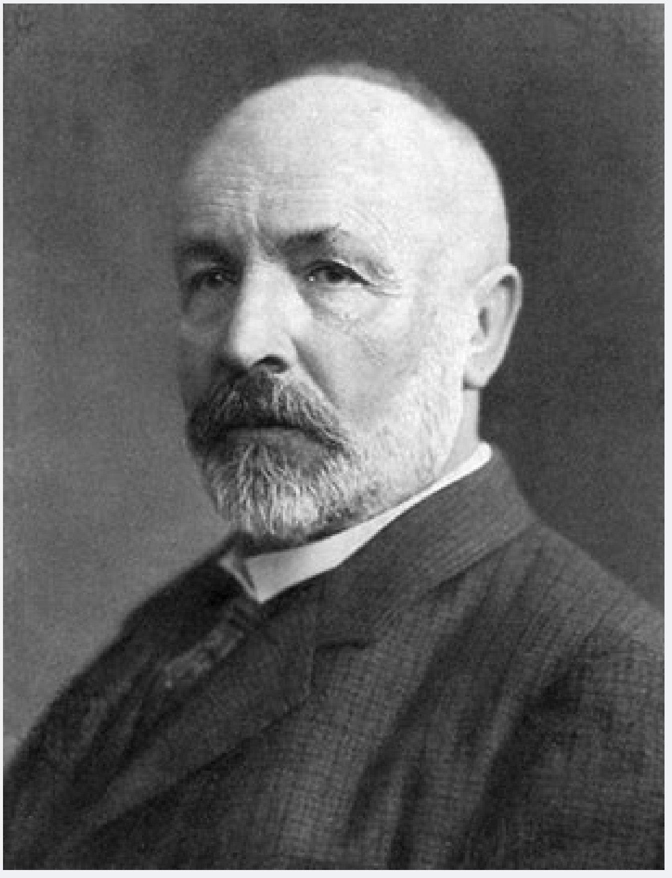

Figure 3: Georg Cantor

Georg Cantor, the father of modern set theory, was born during the year 1845 in Saint Petersburg, Russia and later moved to Germany as a youth. Besides being an accomplished mathematician, he also played the violin. Cantor received his doctoral degree in Mathematics at the age of 22.

In 1870, at the age of 25 he established the uniqueness theorem for trigonometric series. His most significant work happened between 1874 and 1884, when he established the existence of transcendental numbers (also called irrational numbers) and proved that the set of real numbers are uncountably infinite—despite the objections of his former professor Leopold Kronecker.

Cantor published his final treatise on set theory in 1897 at the age of 52, and was awarded the Sylvester Medial from the Royal Society of London in 1904 for his contributions to the field. At the heart of Cantor's work was his goal to solve the continuum problem, which later influenced the works of David Hilbert and Ernst Zermelo.

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Akihiro Kanamori, “Set Theory from Cantor to Cohen,” Editor(s): Dov M. Gabbay, Akihiro Kanamori, John Woods, *Handbook of the History of Logic*, North-Holland, Volume 6, 2012.

Key Terms

- set
- elements
- well-defined set
- empty set
- roster method
- finite set
- infinite set
- natural numbers
- integer
- set-builder notation
- cardinality of a set
- countably infinite
- equal sets
- equivalent sets

Key Concepts

- Identify a set as being a well-defined collection of objects and differentiate between collections that are not well-defined and collections that are sets.
- Represent sets using both the roster or listing method and set builder notation which includes a description of the members of a set.
- In set theory, the following symbols are universally used:

$\mathbb{N}$ - The set of natural numbers, which is the set of all positive counting numbers.

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

$\mathbb{Z}$ - The set of integers, which is the set of all the positive and negative counting numbers and the number zero.

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

$\mathbb{Q}$ - The set of rational numbers or fractions.

$$\mathbb{Q} = \{\frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0\}$$

- Distinguish between finite sets, infinite sets, and the empty set to determine the size or cardinality of a set.
- Distinguish between equal sets which have exactly the same members and equivalent sets that may have different members but must have the same cardinality or size.

Video

- Equal and Equivalent Sets

1.2 Subsets

Figure 4: The players on a soccer team who are actively participating in a game are a subset of the greater set of team members.

Learning Objectives

After completing this section, you should be able to:

1. Represent subsets and proper subsets symbolically.
2. Compute the number of subsets of a set.
3. Apply concepts of subsets and equivalent sets to finite and infinite sets.

The rules of Major League Soccer (MLS) allow each team to have up to 30 players on their team. However, only 18 of these players can be listed on the game day roster, and of the 18 listed, 11 players must be selected to start the game. How the coaches and general managers form the team and choose the starters for each game will determine the success of the team in any given year.

The entire group of 30 players is each team's set. The group of game day players is a **subset** of the team set, and the group of 11 starters is a subset of both the team set and the set of players on the game day roster.

Set A is a subset of set B if every member of set A is also a member of set B. Symbolically, this relationship is written as $A \subseteq B$.

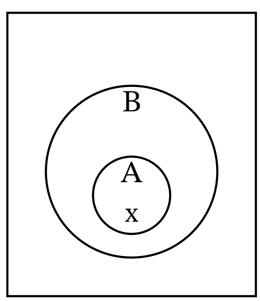
Sets can be related to each other in several different ways: they may not share any members in common, they may share some members in common, or they may share all members in common. In this section, we will explore the way we can select a group of members from the whole set.

> **CHECKPOINT**
> *Every set is also a subset of itself, $B \subseteq B$*

Recall the set of flatware in our kitchen drawer from Section 1.1, $F = \{\text{fork, spoon, knife, meat thermometer, can opener}\}$. Suppose you are preparing to eat dinner, so you pull a fork and a knife from the drawer to set the table. The set $D = \{\text{knife, fork}\}$ is a subset of set F, because every member or element of set D is also a member of set F. More specifically, set D is a **proper subset** of set F, because there are other members of set F not in set D. This is written as $D \subset F$. The only subset of a set that is not a proper subset of the set would be the set itself.

> **CHECKPOINT**
> *The empty set or null set, $\varnothing$, is a proper subset of every set, except itself.*

Graphically, sets are often represented as circles. In the following graphic, set A is represented as a circle completely enclosed inside the circle representing set B, showing that set A is a proper subset of set B. The element x represents an element that is in both set A and set B.



> **CHECKPOINT**
> *While we can list all the subsets of a finite set, it is not possible to list all the possible subsets of an infinite set, as it would take an infinitely long time.*

Example 1 **Listing All the Proper Subsets of a Finite Set**

Set L is a set of reading materials available in a shop at the airport, $L = \{\text{newspaper, magazine, book}\}$. List all the subsets of set L.

Solution

Step 1: It is best to begin with the set itself, as every set is a subset of itself. In our example, the cardinality of set L is $n(L) = 3$. There is only one subset of set L that has the same number of elements of set L : {newspaper, magazine, book}.

Step 2: Next, list all the proper subsets of the set containing $n(L) - 1$ elements. In this case, $3 - 1 = 2$. There are three subsets that each contain two elements: {newspaper, magazine}, {newspaper, book}, and {magazine, book}.

Step 3: Continue this process by listing all the proper subsets of the set containing $n(L) - 2$ elements. In this case, $3 - 2 = 1$. There are three subsets that contain one element: {newspaper}, {magazine}, and {book}.

Step 4: Finally, list the subset containing 0 elements, or the empty set: { }.

Example 2 **Determining Whether a Set Is a Proper Subset**

Consider the set of common political parties in the United States, $P =$ {Democratic, Green, Libertarian, Republican}. Determine if the following sets are proper subsets of P.

1. $M =$ {Democratic, Republican}
2. $G =$ {Green}
3. $V =$ {Republican, Libertarian, Green, Democratic}

Solution

1. M is a proper subset of P, written symbolically as $M \subset P$ because every member of M is a member of set P, but P also contains at least one element that is not in M.
2. G is a single member proper subset of P, written symbolically as $G \subset P$, because Green is a member of set P, but P also contains other members (such as Democratic) that are not in G.
3. V is subset of P because every member of V is also a member of P, but it is not a proper subset of P because there are no members of V that are not also in set P. We can represent the relationship symbolically as $V \subseteq P$, or more precisely, set V is equal to set P, $V = P$.

Example 3 **Expressing the Relationship between Sets Symbolically**

Consider the subsets of a standard deck of cards: $S =$ {spades, hearts, diamonds, clubs} $R =$ {hearts, diamonds} $B =$ {spades, clubs} and $C =$ {clubs}.

Express the relationship between the following sets symbolically.

1. Set S and set B.
2. Set C and set B.
3. Set R and R.

Solution

1. $B \subset S$. B is a proper subset of set S.
2. $C \subset B$. C is a proper subset of set B.
3. $R \subseteq R$ or $R = R$. R is subset of itself, but not a proper subset of itself because R is equal to itself.

Exponential Notation

So far, we have figured out how many subsets exist in a finite set by listing them. Recall that in Example 1, when we listed all the subsets of the three-element set $L = \{\text{newspaper, magazine, book}\}$ we saw that there are eight subsets. In Your Turn 1.11, we discovered that there are four subsets of the two-element subset, $S = \{\text{heads, tails}\}$. A one-element set has two subsets, the empty set and itself. The only subset of the empty set is the empty set itself. But how can we easily figure out the number of subsets in a very large finite set? It turns out that the number of subsets can be found by raising 2 to the number of elements in the set, using **exponential notation** to represent repeated multiplication. For example, the number of subsets of the set $L = \{\text{newspaper, magazine, book}\}$ is equal to $2^3 = 2 \cdot 2 \cdot 2 = 8$. Exponential notation is used to represent repeated multiplication, $b^n = b \cdot b \cdot b \cdot \ldots \cdot b$, where b appears as a factor n times.

FORMULA

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A: Number of Subsets of Set $A = 2^{n(A)}$.

CHECKPOINT

Note that $2^0 = 1$, so this formula works for the empty set, also.

Example 4 **Computing the Number of Subsets of a Set**

Find the number of subsets of each of the following sets.

1. The set of top five scorers of all time in the NBA: $S = \{\text{LeBron James, Kareem Abdul-Jabbar, Karl Malone, Kobe Bryant, Michael Jordan}\}$.
2. The set of the top four bestselling albums of all time: $A = \{\text{Thriller, Hotel California, The Beatles White Album, Led Zepplin IV}\}$.
3. $R = \{\text{Snap, Crackle, Pop}\}$.

Solution

1. $n(S) = 5$. So, the total number of subsets of S is $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.
2. $n(A) = 4$. Therefore, the total number of subsets of A is $2^4 = 16$.
3. $n(R) = 3$. So, the total number of subsets of R is $2^3 = 8$.

Equivalent Subsets

In the early 17th century, the famous astronomer Galileo Galilei found that the set of natural numbers and the subset of the natural numbers consisting of the set of square numbers, n^2, are equivalent. Upon making this discovery, he conjectured that the concepts of less than, greater than, and equal to did not apply to infinite sets.

Sequences and **series** are defined as infinite subsets of the set of natural numbers by forming a relationship between the sequence or series in terms of a natural number, n. For example, the set of even numbers can be defined using set builder notation as $\{a \mid a = 2n$ where n is a natural number$\}$. The formula in this case replaces every natural number with two times the number, resulting in the set of even numbers, $\{2, 4, 6, \ldots\}$. The set of even numbers is also equivalent to the set of natural numbers.

WHO KNEW?

Employment Opportunities

You can make a career out of working with sets. Applications of equivalent sets include relational database design and analysis.

Relational databases that store data are tables of related information. Each row of a table has the same number of columns as every other row in the table; in this way, relational databases are examples of set equivalences for finite sets. In a relational database, a primary key is set up to identify all related information. There is a one-to-one relationship between the primary key and any other information associated with it.

Database design and analysis is a high demand career with a median entry-level salary of about $85,000 per year, according to salary.com.

Example 5 **Writing Equivalent Subsets of an Infinite Set**

Using natural numbers, multiples of 3 are given by the sequence $\{3, 6, 9, \ldots\}$. Write this set using set builder notation by expressing each multiple of 3 using a formula in terms of a natural number, n.

Solution

$\{m \mid m = 3n$ where n is a natural number$\}$ or $\{m \mid m = 3n$ where $n \in \mathrm{N}\}$. In this example, m is a multiple of 3 and n is a natural number. The symbol $\in$ is read as "is a member or element of." Because there is a one-to-one correspondence between the set of multiples of 3 and the natural numbers, the set of multiples of 3 is an equivalent subset of the natural numbers.

Example 6 **Creating Equivalent Subsets of a Finite Set That Are Not Equal**

A fast-food restaurant offers a deal where you can select two options from the following set of four menu items for $6: a chicken sandwich, a fish sandwich, a cheeseburger, or 10 chicken nuggets. Javier and his friend Michael are each purchasing lunch using this deal. Create two equivalent, but not equal, subsets that Javier and Michael could choose to have for lunch.

Solution

The possible two-element subsets are: {chicken sandwich, fish sandwich}, {chicken sandwich, cheeseburger}, {chicken sandwich, chicken nuggets}, {fish sandwich, cheeseburger}, {fish sandwich, chicken nuggets}, and {cheeseburger, chicken nuggets}. One possible solution is that Javier picked the set {chicken sandwich, chicken nuggets}, while Michael chose the {cheeseburger, chicken nuggets}. Because Javier and Michael both picked two items, but not exactly the same two items, these sets are equivalent, but not equal.

Example 7 **Creating Equivalent Subsets of a Finite Set**

A high school volleyball team at a small school consists of the following players: {Angie, Brenda, Colleen, Estella, Maya, Maria, Penny, Shantelle}. Create two possible equivalent starting line-ups of six players that the coach could select for the next game.

Solution

There are actually 28 possible ways that the coach could choose his starting line-up. Two such equivalent subsets are {Angie, Brenda, Maya, Maria, Penny, Shantelle} and {Angie, Brenda, Colleen, Estella, Maria, Shantelle}. Each subset has six members, but they are not identical, so the two sets are equivalent but not equal.

Key Terms

- subset
- proper subset
- equivalent subsets
- exponential notation
  - Every member of a subset of a set is also a member of the set containing it. $A \subseteq B$
- A proper subset of a set does not contain all the members of the set containing it. There is a least one member of set B that is not a member of set A. $A \subset B$
- The number subsets of a finite set A with $n(A)$ members is equal to 2 raised to the $n(A)$ power.
- The empty set is a subset of every set and must be included when listing all the subsets of a set.
- Understand how to create and distinguish between equivalent subsets of finite and infinite sets that are not equal to the original set.

Formulas

The number of subsets of a finite set A is equal to 2 raised to the power of $n(A)$, where $n(A)$ is the number of elements in set A: Number of Subsets of Set $A = 2^{n(A)}$.

1.3 Understanding Venn Diagrams

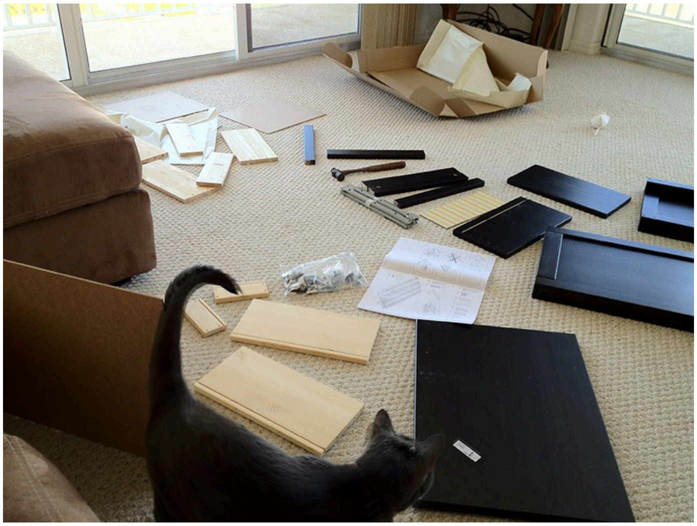


Figure 6: When assembling furniture, instructions with images are easier to follow, just like how set relationships are easier to understand when depicted graphically.

Learning Objectives

After completing this section, you should be able to:

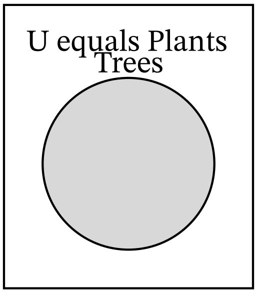
1. Utilize a universal set with two sets to interpret a Venn diagram.
2. Utilize a universal set with two sets to create a Venn diagram.
3. Determine the complement of a set.

Have you ever ordered a new dresser or bookcase that required assembly? When your package arrives you excitedly open it and spread out the pieces. Then you check the assembly guide and verify that you have all the parts required to assemble your new dresser. Now, the work begins. Luckily for you, the assembly guide includes step-by-step instructions with images that show you how to put together your product. If you are really lucky, the manufacturer may even provide a URL or QR code connecting you to an online video that demonstrates the complete assembly process. We can likely all agree that assembly instructions are much easier to follow when they include images or videos, rather than just written directions. The same goes for the relationships between sets.

Interpreting Venn Diagrams

Venn diagrams are the graphical tools or pictures that we use to visualize and understand relationships between sets. Venn diagrams are named after the mathematician John Venn, who first popularized their use in the 1880s. When we use a Venn diagram to visualize the relationships between sets, the entire set of data under consideration is drawn as a rectangle, and subsets of this set are drawn as circles completely contained within the rectangle. The entire set of data under consideration is known as the **universal set**.

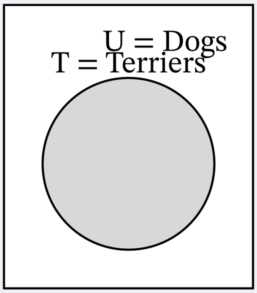
Consider the statement: All trees are plants. This statement expresses the relationship between the set of all plants and the set of all trees. Because every tree is a plant, the set of trees is a subset of the set of plants. To represent this relationship using a Venn diagram, the set of plants will be our universal set and the set of trees will be the subset. Recall that this relationship is expressed symbolically as: $\text{Trees} \subset \text{Plants}$. To create a Venn diagram, first we draw a rectangle and label the universal set "$U = \text{Plants}$." Then we draw a circle within the universal set and label it with the word "Trees."



This section will introduce how to interpret and construct Venn diagrams. In future sections, as we expand our knowledge of relationships between sets, we will also develop our knowledge and use of Venn diagrams to explore how multiple sets can be combined to form new sets.

Example 1 **Interpreting the Relationship between Sets in a Venn Diagram**

Write the relationship between the sets in the following Venn diagram, in words and symbolically.

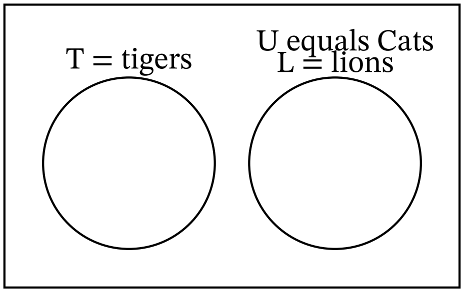


Solution

The set of terriers is a subset of the universal set of dogs. In other words, the Venn diagram depicts the relationship that all terriers are dogs. This is expressed symbolically as $T \subset U$.

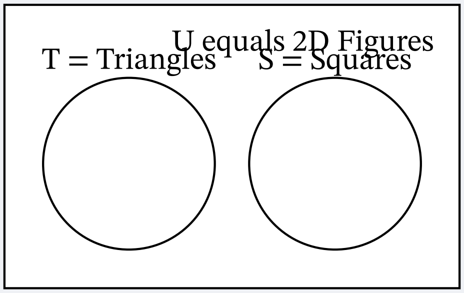
So far, the only relationship we have been considering between two sets is the subset relationship, but sets can be related in other ways. Lions and tigers are both different types of cats, but no lions are tigers, and no tigers are lions. Because the set of all lions and the set of all tigers do not have any members in common, we call these two sets **disjoint sets**, or non-overlapping sets.

Two sets A and B are disjoint sets if they do not share any elements in common. That is, if a is a member of set A, then a is not a member of set B. If b is a member of set B, then b is not a member of set A. To represent the relationship between the set of all cats and the sets of lions and tigers using a Venn diagram, we draw the universal set of cats as a rectangle and then draw a circle for the set of lions and a separate circle for the set of tigers within the rectangle, ensuring that the two circles representing the set of lions and the set of tigers do not touch or overlap in any way.



Example 2 **Describing the Relationship between Sets**

Describe the relationship between the sets in the following Venn diagram.



Solution

The set of triangles and the set of squares are two disjoint subsets of the universal set of two-dimensional figures. The set of triangles does not share any elements in common with the set of squares. No triangles are squares and no squares are triangles, but both squares and triangles are 2D figures.

Creating Venn Diagrams

The main purpose of a Venn diagram is to help you visualize the relationship between sets. As such, it is necessary to be able to draw Venn diagrams from a written or symbolic description of the relationship between sets.

Procedure

To create a Venn diagram:

1. Draw a rectangle to represent the universal set, and label it U = set name.
2. Draw a circle within the rectangle to represent a subset of the universal set and label it with the set name.

CHECKPOINT

If there are multiple disjoint subsets of the universal set, their separate circles should not touch or overlap.

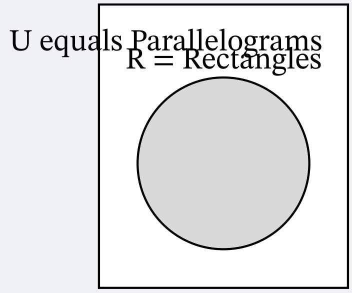
Example 3 **Drawing a Venn Diagram to Represent the Relationship Between Two Sets**

Draw a Venn diagram to represent the relationship between each of the sets.

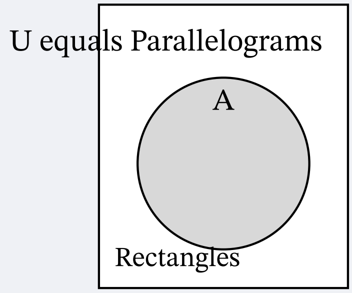
1. All rectangles are parallelograms.
2. All women are people.

Solution

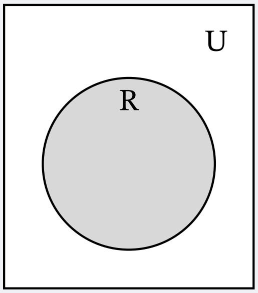
1. The set of rectangles is a subset of the set of parallelograms.
   First, draw a rectangle to represent the universal set and label it with $U =$ Parallelograms, then draw a circle completely within the rectangle, and label it with the name of the set it represents, $R =$ Rectangles.



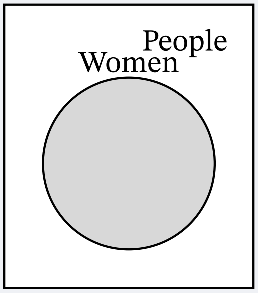
In this example, both letters and names are used to represent the sets involved, but this is not necessary. You may use either letters or names alone, as long as the relationship is clearly depicted in the diagram, as shown below.



or



1. The universal set is the set of people, and the set of all women is a subset of the set of people.



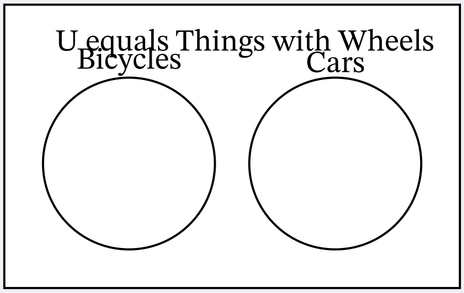
Example 4 **Drawing a Venn Diagram to Represent the Relationship Between Three Sets**

All bicycles and all cars have wheels, but no bicycle is a car. Draw a Venn diagram to represent this relationship.

Solution

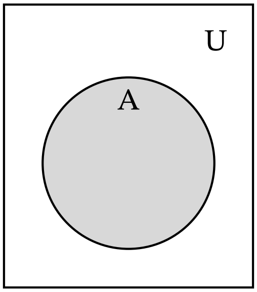
Step 1: The set of bicycles and the set of cars are both subsets of the set of things with wheels. The universal set is the set of things with wheels, so we first draw a rectangle and label it with U = Things with Wheels.

Step 2: Because the set of bicycles and the set of cars do not share any elements in common, these two sets are disjoint and must be drawn as two circles that do not touch or overlap with the universal set.

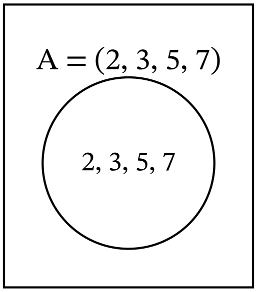


The Complement of a Set

Recall that if set A is a proper subset of set U, the universal set (written symbolically as $A \subset U$), then there is at least one element in set U that is not in set A. The set of all the elements in the universal set U that are not in the subset A is called the **complement** of set A, A'. In set builder notation this is written symbolically as: $A' = \{x \in U \mid x \notin A\}$. The symbol $\in$ is used to represent the phrase, "is a member of," and the symbol $\notin$ is used to represent the phrase, "is not a member of." In the Venn diagram below, the complement of set A is the region that lies outside the circle and inside the rectangle. The universal set U includes all of the elements in set A and all of the elements in the complement of set A, and nothing else.



Consider the set of digit numbers. Let this be our universal set, $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Now, let set A be the subset of U consisting of all the prime numbers in set U, $A = \{2, 3, 5, 7\}$. The complement of set A is $A' = \{0, 1, 4, 6, 8, 9\}$. The following Venn diagram represents this relationship graphically.



Example 5 **Finding the Complement of a Set**

For both of the questions below, A is a proper subset of U.

1. Given the universal set U = {Billie Eilish, Donald Glover, Bruno Mars, Adele, Ed Sheeran} and set A = {Donald Glover, Bruno Mars, Ed Sheeran}, find A'.
2. Given the universal set $U = \{d|d \text{ is a dog}\}$ and $B = \{b \in U|b \text{ is a beagle}\}$, find B'.

Solution

1. The complement of set A is the set of all elements in the universal set U that are not in set A. A' = {Billie Eilish, Adele}.

2. The complement of set B is the set of all dogs that are not beagles. All members of set B' are in the universal set because they are dogs, but they are not in set B, because they are not beagles. This relationship can be expressed in set build notation as follows: $B' = \{\text{All dogs that are not beagles .}\}$, $B' = \{d \in U \mid d \text{ is not a beagle .}\}$, or $B' = \{d \in U \mid d \notin B\}$.

Key Terms

- Venn diagram
- universal set
- disjoint set
- complement of a set

Key Concepts

- A Venn diagram is a graphical representation of the relationship between sets.
- In a Venn diagram, the universal set, U is the largest set under consideration and is drawn as a rectangle. All subsets of the universal set are drawn as circles within this rectangle.
- The complement of set A includes all the members of the universal set that are not in set A. A set and its complement are disjoint sets, they do not share any elements in common.
- To find the complement of set A remove all the elements of set A from the universal set U, the set that includes only the remaining elements is the complement of set A, A'.
- Determine the complement of a set using Venn diagrams, the roster method and set builder notation.

1.4 Set Operations with Two Sets

Figure 18: A large, multigenerational family contains an intersection and a union of sets.

Learning Objectives

After completing this section, you should be able to:

1. Determine the intersection of two sets.
2. Determine the union of two sets.
3. Determine the cardinality of the union of two sets.
4. Apply the concepts of AND and OR to set operations.
5. Draw conclusions from Venn diagrams with two sets.

The movie *Yours, Mine, and Ours* was originally released in 1968 and starred Lucille Ball and Henry Fonda. This movie, which is loosely based on a true story, is about the marriage of Helen, a widow with eight children, and Frank, a widower with ten children, who then have an additional child together. The movie is a comedy that plays on the interpersonal and organizational struggles of feeding, bathing, and clothing twenty people in one household.

If we consider the set of Helen's children and the set of Frank's children, then the child they had together is the intersection of these two sets, and the collection of all their children combined is the union of these two sets. In this section, we will explore the operations of union and intersection as it relates to two sets.

The Intersection of Two Sets

The members that the two sets share in common are included in the **intersection of two sets**. To be in the intersection of two sets, an element must be in both the first set and the second set. In this way, the intersection of two sets is a logical AND statement. Symbolically, A intersection B is written as: $A \cap B$. A intersection B is written in set builder notation as: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Let us look at Helen's and Frank's children from the movie *Yours, Mine, and Ours.* Helen's children consist of the set $H = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph}\}$ and Frank's children are included in the set $F = \{\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph}\}$. H intersection F is the set of children they had together. $H \cap F = \{\text{Joseph}\}$, because Joseph is in both set H and set F.

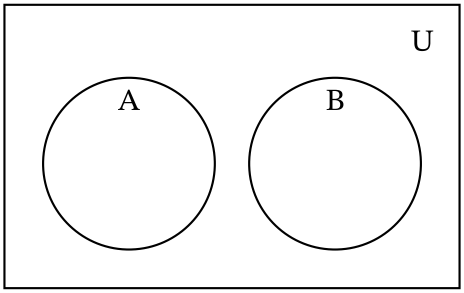
Example 1 **Finding the Intersection of Set A and Set B**

Set $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$. Find A intersection B.

Solution

The intersection of sets A and B include the elements that set A and B have in common: 3, 5, and 7. $A \cap B = \{3, 5, 7\}$.

Notice that if sets A and B are disjoint sets, then they do not share any elements in common, and A intersection B is the empty set, as shown in the Venn diagram below.



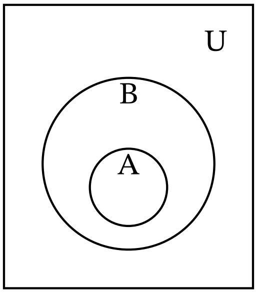
Example 2 **Determining the Intersection of Disjoint Sets**

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cap B$.

Solution

Because sets A and B are disjoint, they do not share any elements in common. So, the intersection of set A and set B is the empty set. $A \cap B = \varnothing$.

Notice that if set A is a subset of set B, then A intersection B is equal to set A, as shown in the Venn diagram below.



Example 3 **Finding the Intersection of a Set and a Subset**

Set $A = \{1, 3, 5, \ldots\}$ and set $B = \mathbb{N} = \{1, 2, 3, \ldots\}$ Find $A \cap B$.

Solution

Because set A is a subset of set B, A intersection B is equal to set A. $A \cap B = A = \{1, 3, 5, \ldots\}$, the set of odd natural numbers.

The Union of Two Sets

Like the union of two families in marriage, the **union of two sets** includes all the members of the first set and all the members of the second set. To be in the union of two sets, an element must be in the first set, the second set, or both. In this way, the union of two sets is a logical inclusive OR statement. Symbolically, A union B is written as: $A \cup B$. A union B is written in set builder notation as: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Let us consider the sets of Helen's and Frank's children from the movie *Yours, Mine, and Ours* again. Helen's children is set $H =$ {Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph} and Frank's children is set $F = \{\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan, Joseph}\}$. The union of these two sets is the collection of all nineteen of their children, $H \cup F = \{\text{Colleen, Nick, Janette, Tommy, Jean, Phillip, Gerald, Theresa, Joseph,}$

$\text{Mike, Rusty, Greg, Rosemary, Loise, Susan, Veronica, Mary, Germaine, Joan}\}$.

Notice, Joseph is in both set H and set F, but he is only one child, so, he is only listed once in the union.

Example 4 **Finding the Union of Sets A and B When A and B Overlap**

Set $A = \{1, 3, 5, 7, 9\}$ and set $B = \{2, 3, 5, 7\}$. Find A union B.

Solution

A union B is the set formed by including all the unique elements in set A, set B, or both sets A and B: $A \cup B = \{1, 3, 5, 7, 9, 2\}$. The first five elements of the union are the five unique elements in set A. Even though 3, 5, and 7 are also members of set B, these elements are only listed one time. Lastly, set B includes the unique element 2, so 2 is also included as part of the union of sets A and B.

When observing the union of sets A and B, notice that both set A and set B are subsets of A union B. Graphically, A union B can be represented in several different ways depending on the members that they have in common. If A and B are disjoint sets, then A union B would be represented with two disjoint circles within the universal set, as shown in the Venn diagram below.

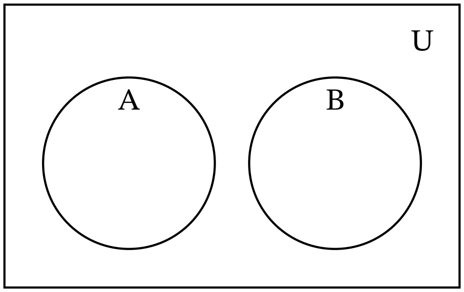
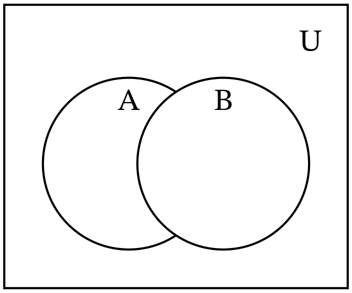
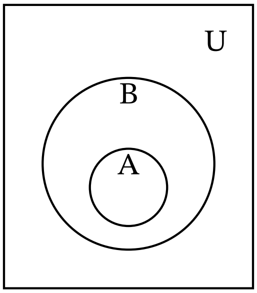


Figure 21: $A \cup B$

If sets A and B share some, but not all, members in common, then the Venn diagram is drawn as two separate circles that overlap.



If every member of set A is also a member of set B, then A is a subset of set B, and A union B would be equal to set B. To draw the Venn diagram, the circle representing set A should be completely enclosed in the circle containing set B.



Example 5 **Finding the Union of Sets A and B When A and B Are Disjoint**

Set $A = \{0, 2, 4, 6, 8\}$ and set $B = \{1, 3, 5, 7, 9\}$. Find $A \cup B$.

Solution

Because sets A and B are disjoint, the union is simply the set containing all the elements in both set A and set B. $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example 6 **Finding the Union of Sets A and B When One Set is a Subset of the Other**

Set $A = \{1, 3, 5, \ldots\}$ and set $B = \mathbb{N} = \{1, 2, 3, \ldots\}$. Find $A \cup B$.

Solution

Because set A is a subset of set B, A union B is equal to set B. $A \cup B = \mathbb{N} = \{1, 2, 3, \ldots\} = B$.

VIDEO

The Basics of Intersection of Sets, Union of Sets and Venn Diagrams

TECH CHECK

Set Operation Practice

Sets Challenge is an application available on both Android and iPhone smartphones that allows you to practice and gain familiarity with the operations of set union, intersection, complement, and difference.

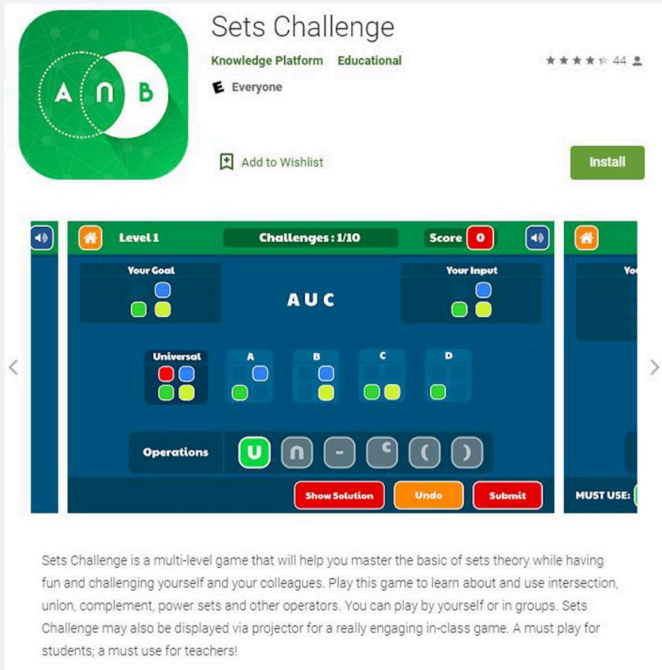


Figure 24: Google Play Store image of Sets Challenge game.

The Sets Challenge application/game uses some notation that differs from the notation covered in the text.

- The complement of set A in this text is written symbolically as A', but the Sets Challenge game uses A^C to represent the complement operation.
- In the text we do not cover set difference between two sets A and B, represented in the game as $A - B$. In the game this operation removes from set A all the elements in $A \cap B$. For example, if set $A = \{a, b, c, d\}$ and set $B = \{b, d, f, h\}$ are subsets of the universal set $U = \{a, b, c, \ldots, z\}$, then $A - B = \{a, b, c, d\} - \{b, d\} = \{a, c\}$, and $B - A = \{b, d, f, h\} - \{b, d\} = \{f, h\}$. There is a project at the end of the chapter to research the set difference operation.

Determining the Cardinality of Two Sets

The **cardinality of the union of two sets** is the total number of elements in the set. Symbolically the cardinality of A union B is written, $n(A \cup B)$. If two sets A and B are disjoint, the cardinality of A union B is the sum of the cardinality of set A and the cardinality of set B. If the two sets intersect, then A intersection B is a subset of both set A and set B. This means that if we add the cardinality of set A and set B, we will have added the number of elements in A intersection B twice, so we must then subtract it once as shown in the formula that follows.

FORMULA

The **cardinality of A union B** is found by adding the number of elements in set A to the number of elements in set B, then subtracting the number of elements in the intersection of set A and set B. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ or $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.

CHECKPOINT

If sets A and B are disjoint, then $n(A \cap B) = n(A \text{ and } B) = 0$ and the formula is still valid, but simplifies to $n(A \cup B) = n(A) + n(B)$.

Example 7 **Determining the Cardinality of the Union of Two Sets**

The number of elements in set A is 10, the number of elements in set B is 20, and the number of elements in A intersection B is 4. Find the number of elements in A union B.

Solution

Using the formula for determining the cardinality of the union of two sets, we can say $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10 + 20 - 4 = 26$.

Example 8 **Determining the Cardinality of the Union of Two Disjoint Sets**

If A and B are disjoint sets and the cardinality of set A is 37 and the cardinality of set B is 43, find the cardinality of A union B.

Solution

To find the cardinality of A union B, apply the formula, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Because sets A and B are disjoint, $A \cap B$ is the empty set, therefore $n(A \cap B) = n(\varnothing) = 0$ and $n(A \cup B) = 37 + 43 - 0 = 80$.

Applying Concepts of “AND” and “OR” to Set Operations

To become a licensed driver, you must pass some form of written test and a road test, along with several other requirements depending on your age. To keep this example simple, let us focus on the road test and the written test. If you pass the written test but fail the road test, you will not receive your license. If you fail the written test, you will not be allowed to take the road test and you will not receive a license to drive. To receive a driver’s license, you must pass the written test AND the road test. For an “AND” statement to be true, both conditions that make up the statement must be true. Similarly, the intersection of two sets A and B is the set of elements that are in both set A and set B. To be a member of A intersection B, an element must be in set A and also must be in set B. The intersection of two sets corresponds to a logical “AND” statement.

The union of two sets is a logical inclusive “OR” statement. Say you are at a birthday party and the host offers Leah, Lenny, Maya, and you some cake or ice cream for dessert. Leah asks for cake, Lenny accepts both cake and ice cream, Maya turns down both, and you choose only ice cream. Leah, Lenny, and you are all having dessert. The “OR” statement is true if at least one of the components is true. Maya is the only one who did not have cake or ice cream; therefore, she did not have dessert and the “OR” statement is false. To be in the union of two sets A and B, an element must be in set A or set B or both set A and set B.

Example 9 **Applying the "AND" or "OR" Operation**

$A = \{0, 3, 6, 9, 12\}, B = \{0, 4, 8, 12, 16\}$, and $C = \{1, 2, 3, 5, 8, 13\}$.

Find the set consisting of elements in:

1. A and B.
2. A or B.
3. A or C.
4. (B and C) or A.

Solution

1. A and $B = A \cap B = \{0, 12\}$, because only the elements 0 and 12 are members of both set A and set B.
2. A or $B = A \cup B = \{0, 3, 4, 6, 8, 9, 12, 16\}$, because the set A or B is the collection of all elements in set A or set B, or both.
3. A or $C = A \cup C = \{0, 1, 2, 3, 5, 6, 8, 9, 12, 13\}$, because the set A or C is the collection of all elements in set A or set C, or both.
4. (B and C) or $A = (B \cap C) \cup A$. Parentheses are evaluated first: (B and C) $= B \cap C = \{8\}$, because the only member that both set B and set C share in common is 8. So, now we need to find $\{8\}$ or $\{0, 3, 6, 9, 12\}$, Because the word translates to the union operation, the problem becomes $\{8\} \cup \{0, 3, 6, 9, 12\}$, which is equal to $\{0, 3, 6, 8, 9, 12\}$.

Example 10 **Determine and Apply the Appropriate Set Operations to Solve the Problem**

Don Woods is serving cake and ice cream at his Juneteenth celebration. The party has a total of 54 guests in attendance. Suppose 30 guests requested cake, 20 guests asked for ice cream, and 12 guests did not have either cake or ice cream.

1. How many guests had cake or ice cream?
2. How many guests had cake and ice cream?

Solution

1. The total number of people at the party is 54, and 12 people did not have cake or ice cream. Recall that the total number of elements in the universal set is always equal to the number of elements in a subset plus the number of elements in the complement of the set, $n(U) = n(A) + n(A')$. That means $54 = n(\text{cake or ice cream}) + n(\text{not (cake or ice cream)})$, or equivalently, $n(\text{cake} \cup \text{ice cream}) = 54 - n\left((\text{cake} \cup \text{ice cream})'\right) = 54 - 12 = 42$. A total of 42 people at the party had cake or ice cream.
2. To determine the number of people who had both cake and ice cream, we need to find the intersection of the set of people who had cake and the set of people who had

ice cream. From Question 1, the number of people who had cake or ice cream is 42. This is the union of the two sets. The formula for the union of two sets is $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Use the information given in the problem and substitute the known values into the formula to solve for the number of people in the intersection: $42 = 30 + 20 - n(A \cap B)$. Adding 30 and 20, the equation simplifies to $42 = 50 - n(\text{cake and ice cream})$. Which means $n(\text{cake and ice cream}) = 50 - 42 = 8$.

WHO KNEW?

The Real Inventor of the Venn Diagram

John Venn, in his writings, references works by both John Boole and Augustus De Morgan, who referred to the circle diagrams commonly used to present logical relationships as Euler's circles. Leonhard Euler's works were published over 100 years prior to Venn's, and Euler may have been influenced by the works of Gottfried Leibniz.

So, why does John Venn get all the credit for these graphical depictions? Venn was the first to formalize the use of these diagrams in his book *Symbolic Logic,* published in 1881. Further, he made significant improvements in their design, including shading to highlight the region of interest. The mathematician C.L. Dodgson, also known as Lewis Carroll, built upon Venn's work by adding an enclosing universal set.

Invention is not necessarily coming up with an initial idea. It is about seeing the potential of an idea and applying it to a new situation.

References:

Margaret E. Baron. "A Note on the Historical Development of Logic Diagrams: Leibniz, Euler and Venn." *The Mathematical Gazette*, vol. 53, no. 384, 1969, pp. 113-125. *JSTOR*, www.jstor.org/stable/3614533. Accessed 15 July 2021.

Deborah Bennett. "Drawing Logical Conclusions." *Math Horizons*, vol. 22, no. 3, 2015, pp. 12-15. *JSTOR*, www.jstor.org/stable/10.4169/mathhorizons.22.3.12. Accessed 15 July 2021.

Drawing Conclusions from a Venn Diagram with Two Sets

All Venn diagrams will display the relationships between the sets, such as subset, intersecting, and/or disjoint. In addition to displaying the relationship between the two sets, there are two main additional details that Venn diagrams can include: the individual members of the sets or the cardinality of each disjoint subset of the universal set.

A Venn diagram with two subsets will partition the universal set into 3 or 4 sections depending on whether they are disjoint or intersecting sets. Recall that the complement of set A, written A', is the set of all elements in the universal set that are not in set A.

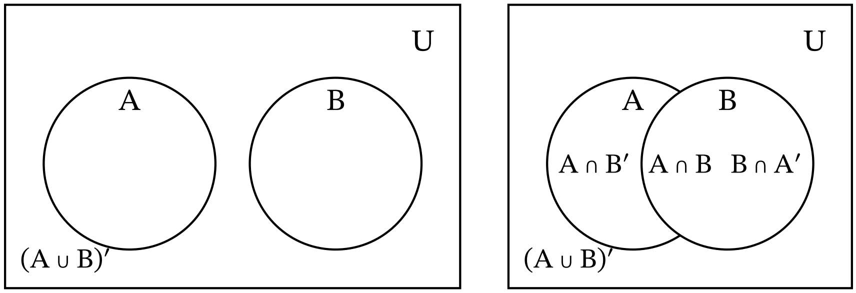
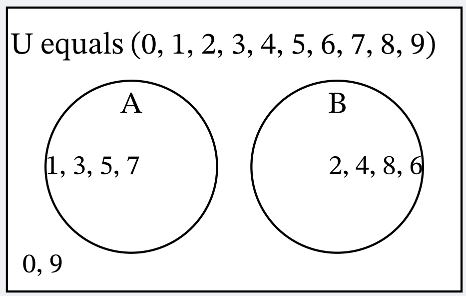


Figure 25: Side-by-side Venn diagrams with disjoint and intersecting sets, respectively.

Example 11 **Using a Venn Diagram to Draw Conclusions about Set Membership**



1. Find $A \cup B$.
2. Find $A \cap B$.
3. Find B'.
4. Find $n(B')$.

Solution

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, because A union B is the collection of all elements in set A or set B or both.
2. Because A and B are disjoint sets, there are no elements that are in both A and B. Therefore, A intersection B is the empty set, $A \cap B = \varnothing$.
3. The complement of set B is the set of all elements in the universal set that are not in set B: $B' = \{0, 1, 3, 5, 7, 9\}$.
4. The cardinality, or number of elements in set B', is $n(B') = 6$.

Example 12 **Using a Venn Diagram to Draw Conclusions about Set Cardinality**

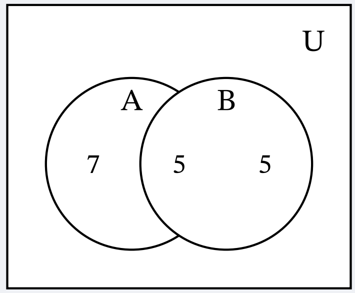


Figure 27: Venn diagram with two intersecting sets and number of elements in each section indicated.

1. Find $n(A \text{ or } B)$.
2. Find $n(A \text{ and } B)$.
3. Find $n(A)$.

Solution

1. The number of elements in A or B is the number of elements in A union B: $n(A \cup B) = n(\{2, 5, 7\}) = 14$.
2. The number of elements in A and B is the number of elements in A intersection B: $n(A \cap B) = 5$.
3. The number of elements in set A is the sum of all the numbers enclosed in the circle representing set A: $n(A) = n(\{7, 5\}) = 12$.

Key Terms

- intersection of two sets
- union of two sets

Key Concepts

- The intersection of two sets, $A \cap B$ is the set of all elements that they have in common. Any member of A intersection B must be is both set A and set B.
- The union of two sets, $A \cup B$, is the collection of all members that are in either in set A, set B or both sets A and B combined.
- Two sets that share at least one element in common, so that they are not disjoint are represented in a Venn Diagram using two circles that overlap.
  - The region of the overlap is the set A intersection B, $A \cap B$.
- The regions that include everything in the circle representing set A or the circle representing set B or their overlap is the set A union B, $A \cup B$.
- Apply knowledge of set union and intersection to determine cardinality and membership using Venn Diagrams, the roster method and set builder notation.

Formulas

The cardinality of A union B : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Video

- The Basics of Intersection of Sets, Union of Sets and Venn Diagrams

1.5 Set Operations with Three Sets

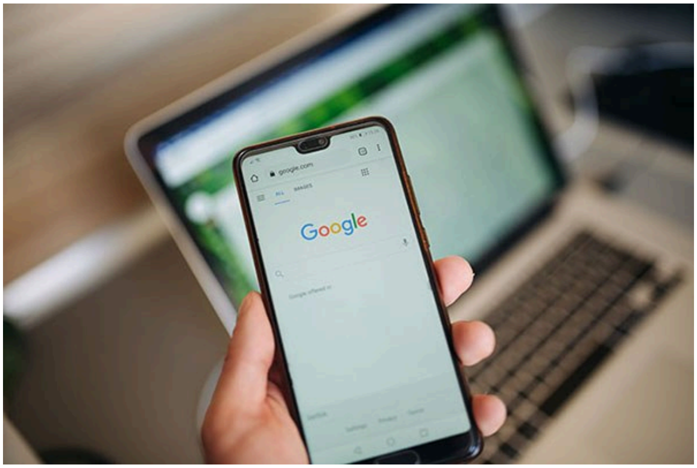


Figure 28: Companies like Google collect data on how you use their services, but the data requires analysis to really mean something.

Learning Objectives

After completing this section, you should be able to:

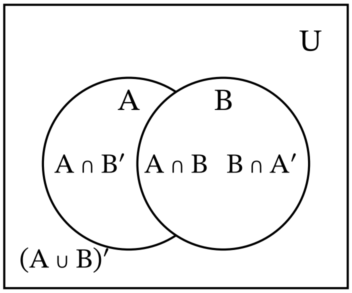
1. Interpret Venn diagrams with three sets.
2. Create Venn diagrams with three sets.
3. Apply set operations to three sets.
4. Prove equality of sets using Venn diagrams.

Have you ever searched for something on the Internet and then soon after started seeing multiple advertisements for that item while browsing other web pages? Large corporations have built their business on data collection and analysis. As we start working with larger data sets, the analysis becomes more complex. In this section, we will extend our knowledge of set relationships by including a third set.

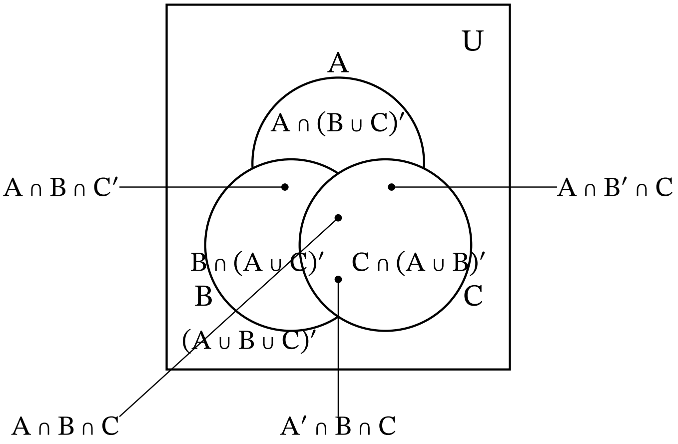
A Venn diagram with two intersecting sets breaks up the universal set into four regions; simply adding one additional set will increase the number of regions to eight, doubling the complexity of the problem.

Venn Diagrams with Three Sets

Below is a Venn diagram with two intersecting sets, which breaks the universal set up into four distinct regions.



Next, we see a **Venn diagram with three intersecting sets**, which breaks up the universal set into eight distinct regions.



TECH CHECK

Shading Venn Diagrams

Venn Diagram is an Android application that allows you to visualize how the sets are related in a Venn diagram by entering expressions and displaying the resulting Venn diagram of the set shaded in gray.

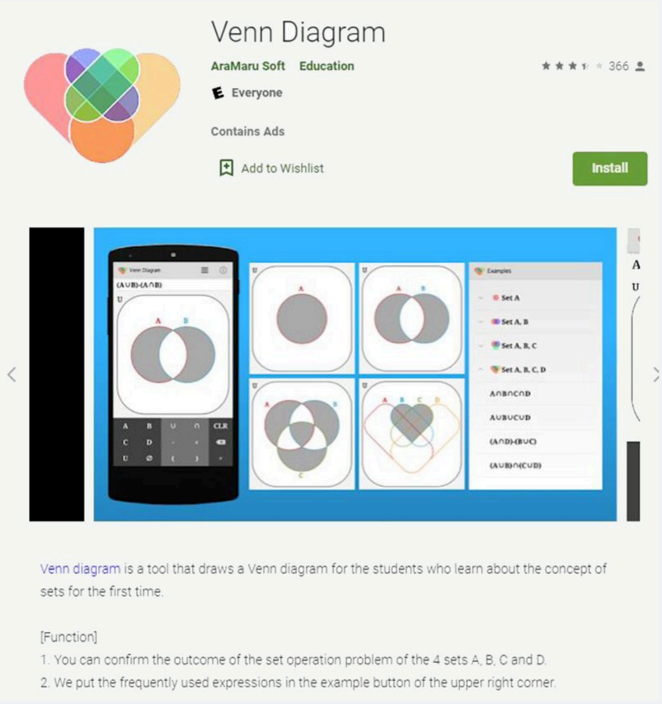


Figure 31: Google Play Store image of Venn Diagram app.

The Venn Diagram application uses some notation that differs from the notation covered in this text.

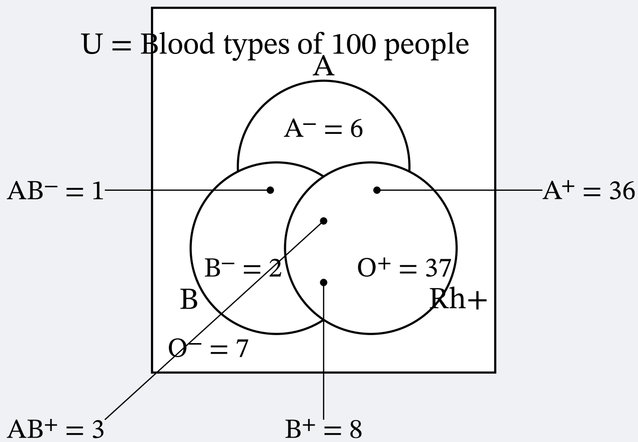
1. The complement of set A in this text is written symbolically as A', but the Venn Diagram app uses A^C to represent the complement operation.
2. The set difference operation, $-$, is available in the Venn Diagram app, although this operation is not covered in the text.

   It is recommended that you explore this application to expand your knowledge of Venn diagrams prior to continuing with the next example.

In the next example, we will explore the three main blood factors, A, B and Rh. The following background information about blood types will help explain the relationships between the sets of blood factors. If an individual has blood factor A or B, those will be included in their blood type. The Rh factor is indicated with a $+$ or a $-$. For example, if a person has all three blood factors, then their blood type would be AB^+. In the Venn diagram, they would be in the intersection of all three sets, $A \cap B \cap Rh^+$. If a person did not have any of these three blood factors, then their blood type would be O^-, and they would be in the set $(A \cup B \cup Rh^+)'$ which is the region outside all three circles.

Example 1 **Interpreting a Venn Diagram with Three Sets**

Use the Venn diagram below, which shows the blood types of 100 people who donated blood at a local clinic, to answer the following questions.



1. How many people with a type A blood factor donated blood?
2. Julio has blood type B^+. If he needs to have surgery that requires a blood transfusion, he can accept blood from anyone who does not have a type A blood factor. How many people donated blood that Julio can accept?
3. How many people who donated blood do not have the Rh^+ blood factor?
4. How many people had type A and type B blood?

Solution

1. The number of people who donated blood with a type A blood factor will include the sum of all the values included in the A circle. It will be the union of sets A^-, A^+, AB^- and AB^+. $n(A) = n(A^-) + n(A^+) + n(AB^-) + n(AB^+) = 6 + 36 + 1 + 3 = 46$.
2. In part 1, it was determined that the number of donors with a type A blood factor is 46. To determine the number of people who did not have a type A blood factor, use the following property, A' union is equal to U, which means $n(A) + n(A') = n(U)$, and $n(A') = n(U) - n(A) = 100 - 46 = 54$. Thus, 54 people donated blood that Julio can accept.
3. This would be everyone outside the Rh^+ circle, or everyone with a negative Rh factor, $n(Rh^-) = n(O^-) + n(A^-) + n(AB^-) + n(B^-) = 7 + 6 + 1 + 2 = 16$.
4. To have both blood type A and blood type B, a person would need to be in the intersection of sets A and B. The two circles overlap in the regions labeled AB^- and AB^+. Add up the number of people in these two regions to get the total: $1 + 3 = 4$. This can be written symbolically as $n(A \text{ and } B) = n(A \cap B) = n(AB^-) + n(AB^+) = 1 + 3 = 4$.

WHO KNEW?

Blood Types

Most people know their main blood type of A, B, AB, or O and whether they are Rh^+ or Rh^-, but did you know that the International Society of Blood Transfusion recognizes twenty-eight additional blood types that have important implications for organ transplants and successful pregnancy? For more information, check out this article:

Blood mystery solved: Two new blood types identified

Creating Venn Diagrams with Three Sets

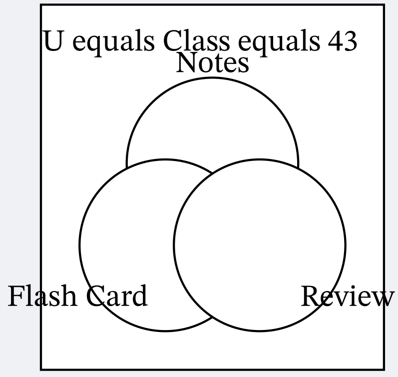
In general, when creating Venn diagrams from data involving three subsets of a universal set, the strategy is to work from the inside out. Start with the intersection of the three sets, then address the regions that involve the intersection of two sets. Next, complete the regions that involve a single set, and finally address the region in the universal set that does not intersect with any of the three sets. This method can be extended to any number of sets. The key is to start with the region involving the most overlap, working your way from the center out.

Example 2 **Creating a Venn Diagram with Three Sets**

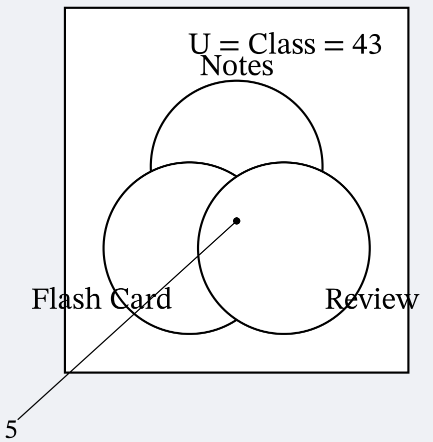
A teacher surveyed her class of 43 students to find out how they prepared for their last test. She found that 24 students made flash cards, 14 studied their notes, and 27 completed the review assignment. Of the entire class of 43 students, 12 completed the review and made flash cards, nine completed the review and studied their notes, and seven made flash cards and studied their notes, while only five students completed all three of these tasks. The remaining students did not do any of these tasks. Create a Venn diagram with subsets labeled: "Notes," "Flash Cards," and "Review" to represent how the students prepared for the test.

Solution

Step 1: First, draw a Venn diagram with three intersecting circles to represent the three intersecting sets: Notes, Flash Cards, and Review. Label the universal set with the cardinality of the class.

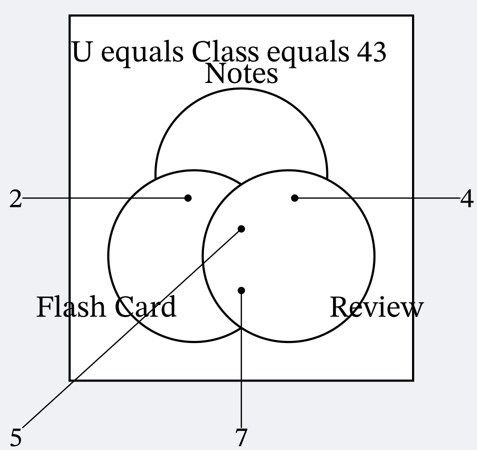


Step 2: Next, in the region where all three sets intersect, enter the number of students who completed all three tasks.



Step 3: Next, calculate the value and label the three sections where just two sets overlap.

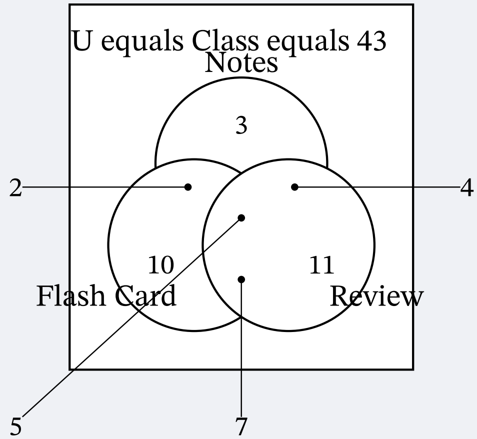
1. **Review and flash card overlap**. A total of 12 students completed the review and made flash cards, but five of these twelve students did all three tasks, so we need to subtract: $12 - 5 = 7$. This is the value for the region where the flash card set intersects with the review set.
2. **Review and notes overlap**. A total of 9 students completed the review and studied their notes, but again, five of these nine students completed all three tasks. So, we subtract: $9 - 5 = 4$. This is the value for the region where the review set intersects with the notes set.
3. **Flash card and notes overlap**. A total of 7 students made flash cards and studied their notes; subtracting the five students that did all three tasks from this number leaves 2 students who only studied their notes and made flash cards. Add these values to the Venn diagram.



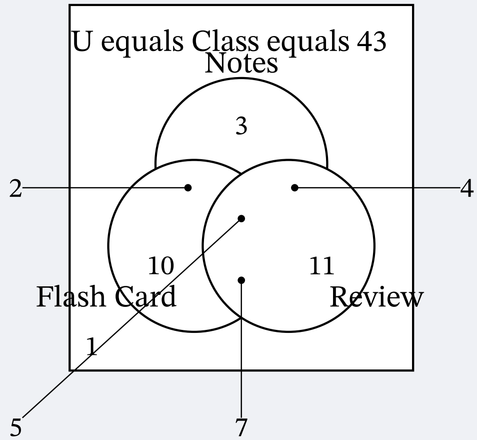
Step 4: Now, repeat this process to find the number of students who only completed one of these three tasks.

1. A total of 24 students completed flash cards, but we have already accounted for $2 + 5 + 7 = 14$ of these. Thus, $24 - 14 = 10$ students who just made flash cards.

4. A total of 14 students studied their notes, but we have already accounted for $4 + 5 + 2 = 11$ of these. Thus, $14 - 11 = 3$ students only studied their notes.
5. A total of 27 students completed the review assignment, but we have already accounted for $4 + 5 + 7 = 16$ of these, which means $27 - 16 = 11$ students only completed the review assignment.
6. Add these values to the Venn diagram.



Step 5: Finally, compute how many students did not do any of these three tasks. To do this, we add together each value that we have already calculated for the separate and intersecting sections of our three sets: $3 + 2 + 4 + 5 + 10 + 7 + 11 = 42$. Because there 43 students in the class, and $43 - 42 = 1$, this means only one student did not complete any of these tasks to prepare for the test. Record this value somewhere in the rectangle, but outside of all the circles, to complete the Venn diagram.



Applying Set Operations to Three Sets

Set operations are applied between two sets at a time. Parentheses indicate which operation should be performed first. As with numbers, the inner most parentheses are applied first. Next, find the complement of any sets, then perform any union or intersections that remain.

Example 3 **Applying Set Operations to Three Sets**

Perform the set operations as indicated on the following sets: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{0, 2, 4, 6, 8, 10, 12\}$, and $C = \{0, 3, 6, 9, 12\}$.

1. Find $(A \cap B) \cap C$.
2. Find $A \cap (B \cup C)$.
3. Find $(A \cap B) \cup C'$.

Solution

1. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$, the elements common to both A and B. $(A \cap B) \cap C = \{0, 2, 4, 6\} \cap \{0, 3, 6, 9, 12\} = \{0, 6\}$, because the only elements that are in both sets are 0 and 6.
2. Parentheses first, B union C equals $B \cup C = \{0, 2, 3, 4, 6, 8, 9, 10, 12\}$, the collection of all elements in set B or set C or both. $A \cap (B \cup C) = \{0, 1, 2, 3, 4, 5, 6\} \cap \{0, 2, 3, 4, 6, 8, 9, 10, 12\} = \{0, 2, 3, 4, 6\}$, because the intersection of these two sets is the set of elements that are common to both sets.
3. Parentheses first, A intersection B equals $A \cap B = \{0, 2, 4, 6\}$. Next, find C'. The complement of set C is the set of elements in the universal set U that are not in set C. $C' = \{1, 2, 4, 5, 7, 8, 10, 11\}$. Finally, find $(A \cap B) \cup C' = \{0, 2, 4, 6\} \cup \{1, 2, 4, 5, 7, 8, 10, 11\} = \{0, 1, 2, 4, 5, 6, 7, 8, 10, 11\}$.

Notice that the answers to the Your Turn are the same as those in the Example. This is not a coincidence. The following equivalences hold true for sets:

- $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$. These are the associative property for set intersection and set union.
- $A \cap B = B \cap A$ and $A \cup B = B \cup A$. These are the commutative property for set intersection and set union.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. These are the distributive property for sets over union and intersection, respectively.

Proving Equality of Sets Using Venn Diagrams

To prove set equality using Venn diagrams, the strategy is to draw a Venn diagram to represent each side of the equality, then look at the resulting diagrams to see if the regions under consideration are identical.

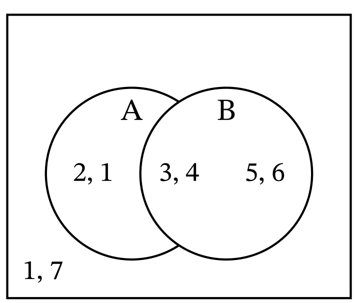
Augustus De Morgan was an English mathematician known for his contributions to set theory and logic. De Morgan's law for set complement over union states that $(A \cup B)' = A' \cap B'$. In the next example, we will use Venn diagrams to prove De Morgan's law for set complement over union is true. But before we begin, let us confirm De Morgan's law works for a specific example. While showing something is true for one specific example is not a proof, it will provide us with some reason to believe that it may be true for all cases.

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 3, 4\}$, and $B = \{3, 4, 5, 6\}$. We will use these sets in the equation $(A \cup B)' = A' \cap B'$. To begin, find the value of the set defined by each side of the equation.

Step 1: $A \cup B$ is the collection of all unique elements in set A or set B or both. $A \cup B = \{2, 3, 4, 5, 6\}$. The complement of A union B, $(A \cup B)'$, is the set of all elements in the universal set that are not in $A \cup B$. So, the left side the equation $(A \cup B)'$ is equal to the set $\{1, 7\}$.

Step 2: The right side of the equation is $A' \cap B'$. A' is the set of all members of the universal set U that are not in set A. $A' = \{1, 5, 6, 7\}$. Similarly, $B' = \{1, 2, 7\}$.

Step 3: Finally, $A' \cap B'$ is the set of all elements that are in both A' and B'. The numbers 1 and 7 are common to both sets, therefore, $A' \cap B' = \{1, 7\}$. Because, $\{1, 7\} = \{1, 7\}$ we have demonstrated that De Morgan's law for set complement over union works for this particular example. The Venn diagram below depicts this relationship.

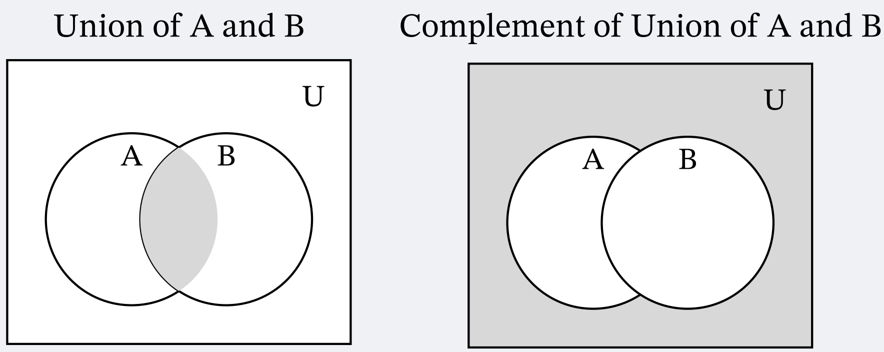


Example 4 **Proving De Morgan's Law for Set Complement over Union Using a Venn Diagram**

De Morgan's Law for the complement of the union of two sets A and B states that: $(A \cup B)' = A' \cap B'$. Use a Venn diagram to prove that De Morgan's Law is true.

Solution

Step 1: First, draw a Venn diagram representing the left side of the equality. The regions of interest are shaded to highlight the sets of interest. $A \cup B$ is shaded on the left, and $(A \cup B)'$ is shaded on the right.



Step 2: Next, draw a Venn diagram to represent the right side of the equation. A' is shaded and B' is shaded. Because A' and B' mix to form $A' \cap B'$ is also shaded.

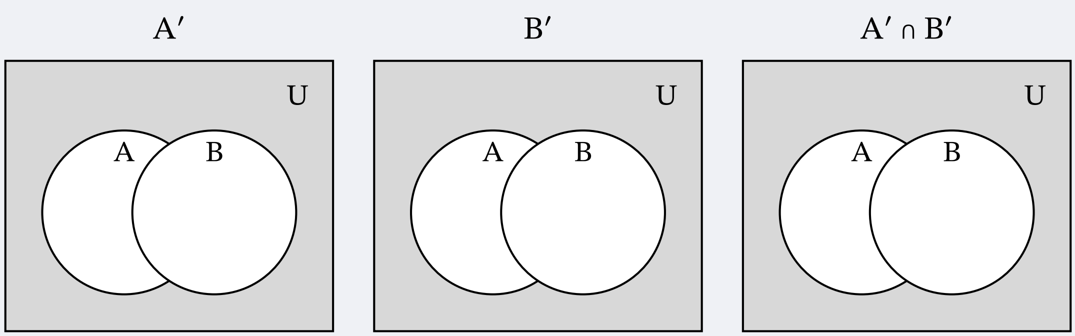


Figure 40: Venn diagram of intersection of the complement of two sets.

Step 3: Verify the conclusion. Because the shaded region in the Venn diagram for $(A \cup B)'$ matches the shaded region in the Venn diagram for $A' \cap B'$, the two sides of the equation are equal, and the statement is true. This completes the proof that De Morgan's law is valid.

Key Concepts

- A Venn diagram with two overlapping sets breaks the universal set up into four distinct regions. When a third overlapping set is added the Venn diagram is broken up into eight distinct regions.
- Analyze, interpret, and create Venn diagrams involving three overlapping sets.
  - Including the blood factors: A, B and Rh

- To find unions and intersections.
- To find cardinality of both unions and intersections.
- When performing set operations with three or more sets, the order of operations is inner most parentheses first, then fine the complement of any sets, then perform any union or intersection operations that remain.
- To prove set equality using Venn diagrams the strategy is to draw a Venn diagram to represent each side of the equality or equation, then look at the resulting diagrams to see if the regions under consideration are identical. If they regions are identical the equation represents a true statement, otherwise it is not true.

Projects

Cardinality of Infinite Sets

In set theory, it has been shown that the set of irrational numbers has a cardinality greater than the set of natural numbers. That is, the set of irrational numbers is so large that it is uncountably infinite.

1. Perform a search with the phrase, "Who first proved that the real numbers are uncountable?"
2. Who first proved that the real numbers are uncountable?
3. What was the significance of this proof to the development of set theory and by extension other fields of mathematics?
4. Recent discoveries in the field of set theory include the solution to a 70-year-old problem previously thought to be unprovable. To learn more read this article:
    1. What does it mean for two infinite sets to have the same size?
5. The real numbers are sometimes referred to as what?
6. Summarize your understanding of the problem known as the "Continuum Hypothesis."
7. Malliaris and Shelah's proof of this 70-year-old problem is opening up investigation in what two fields of mathematics?
8. Summarize your understanding of infinity.
    1. Define what it means to be infinite.
9. Explain the difference between countable and uncountable sets.
10. Research the difference between a discrete set and a continuous set, then summarize your findings.

Set Notation

In arithmetic, the operation of addition is represented by the plus sign, $+$, but multiplication is represented in multiple ways, including $\cdot, \times, *$, and parentheses, such as $5(3)$. Several set operations also are written in different forms based on the preferences of the mathematician and often their publisher.

1. Search for "Set Complement" on the internet and list at least three ways to represent the complement of a set.
2. Both the Set Challenge and Venn Diagram smartphone apps highlighted in the Tech Check sections have an operation for set difference. List at least two ways to represent set difference and provide a verbal description of how to calculate the difference between two sets A and B.
3. When researching possible Venn diagram applications, the Greek letter delta, Δ appeared as a symbol for a set operator. List at least one other symbol used for this same operation.
4. Search for "List of possible set operations and their symbols." Find and select two symbols that were not presented in this chapter.

The Real Number System

The set of real numbers and their properties are studied in elementary school today, but how did the number system evolve? The idea of natural numbers or counting numbers surfaced prior to written words, as evidenced by tally marks in cave writing. Create a timeline for significant contributions to the real number system.

1. Use the following phrase to search online for information on the origins of the number zero: "History of the number zero." Then, record significant dates for the invention and common use of the number zero on your timeline.
2. Find out who is credited for discovering that the $\sqrt{2}$ is irrational and add this information to your timeline. Hint: Search for, "Who was the first to discover irrational numbers?"
3. Research Georg Cantor's contribution to the representation of real numbers as a continuum and add this to your timeline.
4. Research Ernst Zermelo's contribution to the real number system and add this to your timeline.